Inverse spectral problem on discrete graphs

Emilia Blåsten

LUT University

Seminars on Inverse Problems

Theory and Applications

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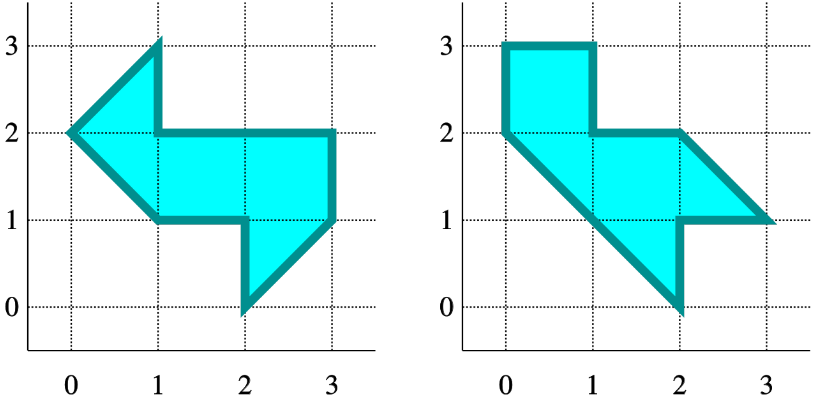
Collaborators: Pavel Exner, Hiroshi Isozaki, Matti Lassas, Jinpeng Lu

Based on:

- B., Isozaki, Lassas, & Lu (2023). Gelfand's inverse problem for the graph laplacian. Journal of Spectral Theory, 13(1), 1–45. http://dx.doi.org/10.4171/jst/455
- B., Isozaki, Lassas, & Lu (2023). Inverse problems for discrete heat equations and random walks for a class of graphs. SIAM Journal on Discrete Mathematics, 37(2), 831–863. http://dx.doi.org/10.1137/21m1439936
- B., Exner, Isozaki, Lassas, & Lu (2024). Inverse problems for locally perturbed lattices – discrete Hamiltonian and quantum graph. To appear in Annales Henri Lebesgues.

Can one hear the shape of a drum?

Mark Kac 1966 & Gordon, Webb, Wolpert 1992



Drumhead shapes whose vibration frequencies are the same. Same Dirichlet Laplacian eigenvalues, different domains.

An inverse spectral problem for (continuous) manifolds

Let M be a Riemannian manifold with boundary ∂M and metric g,

$$\Delta_g u = \sum_{j,k=1}^{n} |g(x)|^{-1/2} \frac{\partial}{\partial x^j}\left(|g(x)|^{1/2} g^{jk}(x) \frac{\partial}{\partial x^k} u(x)\right). \quad (1)$$

The eigenvalues λ_j and orthonormal eigenfunctions $\varphi_j(x)$ satisfy

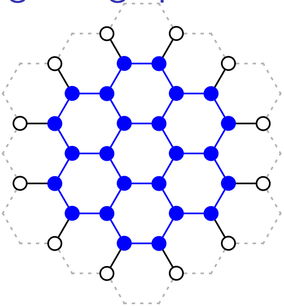
$$\begin{aligned} (-\Delta_g + q(x))\varphi_j(x) &= \lambda_j \varphi_j(x), && \text{for } x \in M, \\ \partial_\nu \varphi_j(x) &= 0 && \text{for } x \in \partial M. \end{aligned}$$

Inverse problem: Suppose we are given the boundary spectral data

$$\Big(\partial M, \, (\lambda_j, \varphi_j|_{\partial M})_{j=1,2,\dots}\Big) \quad (2)$$

Can we determine (M, g) and q?

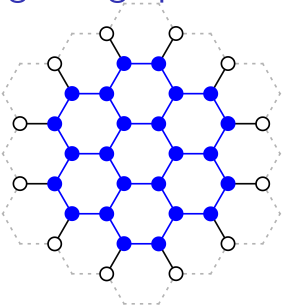
Weighted graph with boundary



We consider a finite graph with vertices $X = G \cup B$ and edges E. We call G the interior nodes and $B = \partial G$ the boundary nodes.

Weights: g_{xy} for $(x, y) \in E$ and μ_x for $x \in X$.

Weighted graph with boundary



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Given $u : G \cup \partial G \to \mathbb{R}$ define

$$\big(\Delta_G u\big)(x) = \frac{1}{\mu_x} \sum_{y \sim x} g_{xy}\big(u(y) - u(x)\big), \quad x \in G, \tag{3}$$

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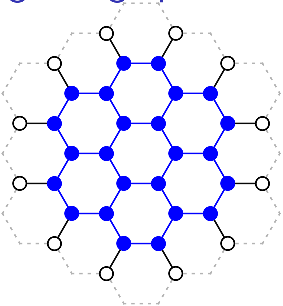

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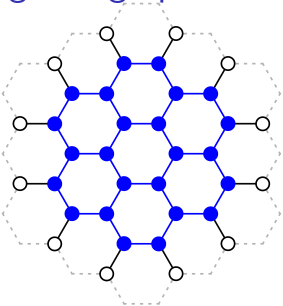

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Combinatorical Laplacian: $g_{xy} = 1$ and $\mu_x = 1$.

Quantum graph vs discrete graph

On quantum graphs:

$$\left(-\frac{d^2}{dx^2} + V_e(x) - \lambda\right)\underline{u_e} = 0, \quad \text{on all edges}$$

$$\underline{u} = f, \quad \text{on graph boundary}$$

$$\sum_{v \in e} \underline{u_e}'(v) = C_v \underline{u}(v), \quad \text{on all vertices } v$$

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On discrete graphs:

$$-\frac{1}{d_v}\sum_{w\sim v}\frac{1}{\phi_e(w,\lambda)}u(w) + \left(\frac{1}{d_v}\sum_{w\sim v}\frac{\phi_e'(w,\lambda)}{\phi_e(w,\lambda)} + \frac{C_v}{d_v}\right)u(v) = 0, \quad v \in G$$

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$\underline{u} = (\underline{u_e})_{e\in E}$ is the solution in the quantum graph iff u is the solution in the discrete graph. ϕ_e is a type of eigenfunction on the edge e.

The inverse spectral problem for a graph

Let $(G \cup \partial G, E)$ be a graph with weights g and μ. Let $q : G \to \mathbb{R}$ be a potential function.

The eigenvalues λ_j and orthonormal eigenfunctions $\varphi_j(x)$ satisfy

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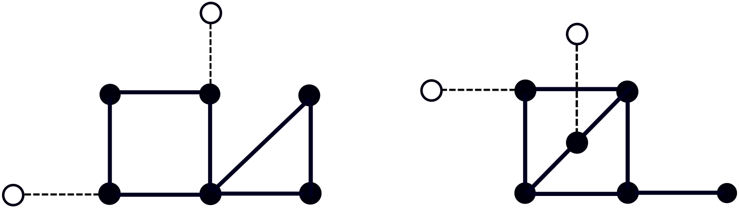
Inverse problem: Suppose we are given the boundary spectral data

$$\Big(\partial G, \, (\lambda_j, \varphi_j|_{\partial G})_{j=1,2,\ldots,|G|}\Big) \tag{5}$$

Can we determine $(G \cup \partial G, E)$ and g, μ and q on G?

Counterexample for the unique solvability of IP

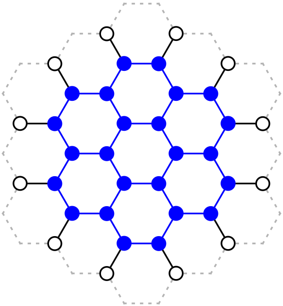


Two graphs on which the combinatorial Laplacian has the same eigenvalues and the boundary values of the eigenfunctions. The white vertices are boundary nodes and the black vertices are interior nodes (our heat equation paper 2023).

Definition (Paths and metric)

Let $x, y \in G \cup \partial G$. A *path* from x to y is a sequence of vertices $v_0, v_1, \ldots, v_J$ such that $v_0 = x$, $v_J = y$ and $v_j \sim v_{j+1}$. The length of the path is J.

The *distance* $d(x, y)$ is the minimal length of a path connecting x and y



Definition (Extreme point of a set)

Let $S \subset G$. We say a point $x_0 \in S$ is an *extreme point* of S, if there exists $z \in \partial G$ such that x_0 is the unique nearest point in S from z.

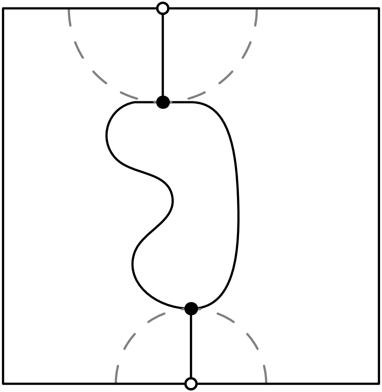


Figure: Any compact subset of the unit square which has at least two points has at least two extreme points.

Two-points condition

We impose the following assumptions on the graph $(G, \partial G, E)$.

(1) For any subset $S \subset G$ with at least 2 points, there exist at least two extreme points.

(2) Each boundary vertex is connected to only one interior vertex.

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This condition is valid for trees, when all vertices of degree one are boundary nodes, and for perturbations of the standard lattices. (Triangular lattice requires (2'))

Theorem (B.-Isozaki-Lassas-Lu 2023)

Let $(G, \partial G, E, \mu, g)$ and $(G', \partial G', E', \mu', g')$ satisfy the Two-points condition. Let q, q' be potential functions on G, G'.

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Assume that ∂G *and* $\partial G'$ *can be identified using a bijective map and the boundary spectral data of these graphs are the same.*

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Assume that ∂G and $\partial G'$ can be identified using a bijective map and the boundary spectral data of these graphs are the same.

Then there is a bijection $\Phi : G \cup \partial G \to G' \cup \partial G'$ such that

$$x_1 \sim x_2 \text{ if and only if } \Phi(x_1) \sim' \Phi(x_2).$$

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Moreover, if we use Φ to identify the graphs G and G', then

(1) *If $\mu = \mu'$, then $g = g'$ and $q = q'$.*

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In other words: The observations at the boundary nodes are enough to determine the structure in the interior of the graph, even when all of the interior nodes G and the edges E are unknown.

Any way for testing two-points condition???

Theorem (B.-Isozaki-Lassas-Lu 2023)

If on the graph $(G, \partial G, E)$ there exists $h : G \cup \partial G \to \mathbb{R}$ such that

1. *If $x \sim y$ then $|h(x) - h(y)| \leqslant 1$,*
2. *$|N_\pm(x)| = 1$ for all $x \in G$, and $|N_\pm(z)| \leqslant 1$ for all $z \in \partial G$, where*

$$N_+(x) = \{y \in G \cup \partial G : \ y \sim x \, , \, h(y) = h(x) + 1\},$$
$$N_-(x) = \{y \in G \cup \partial G : \ y \sim x \, , \, h(y) = h(x) - 1\},$$

then $(G, \partial G, E)$ satisfy the Two-Points Condition.

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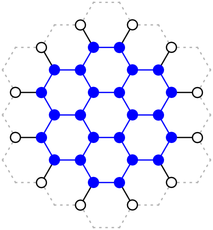
then $(G, \partial G, E)$ *satisfy the Two-Points Condition.*
We call $N_+(x)$ *the discrete gradient of* h *at* x, *and* $N_-(x)$ *the discrete gradient of* $-h$ *at* x.



Graphs satisfying the Two Points Condition

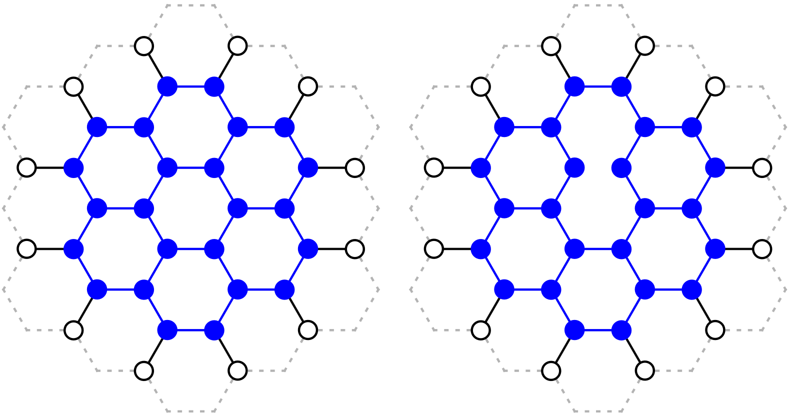


Figure: Finite hexagonal lattice. The white vertices are considered to be the boundary vertices for the set of the blue (interior) vertices. Also, any horizontal edges can be removed.

Graphs satisfying the Two Points Condition

Following graphs satisfy the Two Points Condition when suitable connections to the boundary vertices are removed.
The white vertices are the boundary vertices; the blue vertices are the interior vertices.

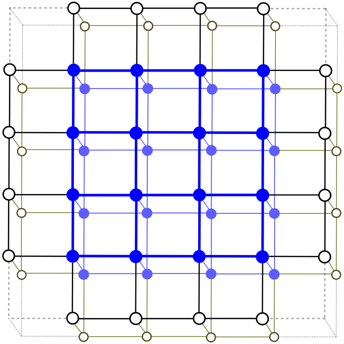


Figure: Finite two-level square ladder, made out of two layers of square lattices.

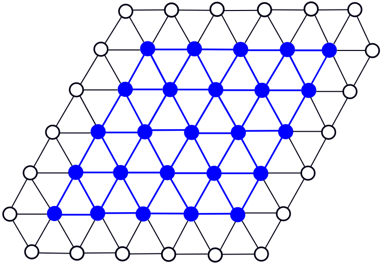


Figure: Finite triangular lattice.

Preparing for the proof

Theorem

Boundary spectral data determines the structure (which points connected to which).

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Tools and interesting facts:

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Tools and interesting facts:

- (Operators) With $x \in G$, $z \in \partial G$

$$\big(\Delta_G u\big)(x) = \frac{1}{\mu_x} \sum_{y \sim x} g_{xy} \big(u(y) - u(x)\big)$$

$$\big(\partial_\nu u\big)(z) = \frac{1}{\mu_z} \sum_{x \sim z,\ x \in G} g_{xz} \big(u(x) - u(z)\big)$$

- (Green's formula) Given $u, v \colon G \cup \partial G \to \mathbb{R}$

$$\sum_G \mu\big(u_1 \Delta_G u_2 - u_2 \Delta_G u_1\big) = \sum_{\partial G} \mu\big(u_2 \partial_\nu u_1 - u_1 \partial_\nu u_2\big)$$

More tools and interesting facts

- (Eigenfunction behaviour) If the graph satisfies the two-points condition then eigenfunctions of $-\Delta_G + q$ cannot have both $u = 0$ and $\partial_\nu u = 0$ on ∂G.
- (N-D $\leftrightarrow$ spectral data) If $(-\Delta_G + q)u_\lambda^f = \lambda u_\lambda^f$ on G and $\partial_\nu u_\lambda^f = f$ on ∂G then

$$u_\lambda^f - w^f = -\sum_{k=1}^{N} \frac{1}{\lambda - \lambda_k} \left(\sum_{z \in \partial G} \mu_z \phi_z(z) f(z) \right) - \sum_{k=1}^{N} \left\langle w^f, \phi_k \right\rangle_{L^2(G)} \phi_k$$

  for any $w^f : G \cup \partial G \to \mathbb{R}$ with $\partial_\nu w^f = f$ on ∂G. Poles of $\Lambda_\lambda f = u_\lambda^f$ given spectrum. Boundary values follow from residues.

Topology from spectral data: proof idea

Lemma

Suppose W is the initial value of some wave u satisfying the wave equation

$$\begin{cases} D_{tt}u(x,t) - \Delta_G u(x,t) + q(x)u(x,t) = 0, & x \in G,\ t \geqslant 1, \\ \partial_\nu u(x,t) = 0, & x \in \partial G,\ t \geqslant 0, \\ D_t u(x,0) = 0, & x \in G, \\ u(x,0) = W(x), & x \in G \cup \partial G, \end{cases} \tag{6}$$

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Here

$$\widehat{W}(j) := \langle W, \varphi_j \rangle := \sum_{x \in G} \mu_x W(x) \varphi_j(x).$$

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Then the boundary spectral data determines the Fourier coefficients of elements of $\mathcal{A}_0$, i.e.

$$\widehat{\mathcal{A}_0} = \{\widehat{W_x} : x \in G\}$$

is determined.

Topology from spectral data: edge determination

Lemma

Let $\mathbb{G}$ be a finite weighted graph with boundary satisfying the two-point condition, and $x, y \in G$. Then $x \sim y$ if and only if

$$\min \left\{ t \in \mathbb{N} \,\middle|\, \langle u^{W_x}(\cdot, t), W_y \rangle \neq 0 \right\} = 2. \tag{7}$$

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We can now identify points $x, y \in G$ with Fourier coefficients $\widehat{W_x}, \widehat{W_y}$ of single point initial values W_x, W_y.

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The boundary spectral data determines if $x \sim y$.

Coefficient determination

For (i), i.e. μ is known

Coefficient determination

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whose LHS is uniquely determined by the boundary spectral data:

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$$\mu_x = \langle W_x, \varphi_0\rangle^2 c^{-2} \tag{9}$$

is determined by the boundary spectral data.

Thank you!