Scattering from corners and other singularities

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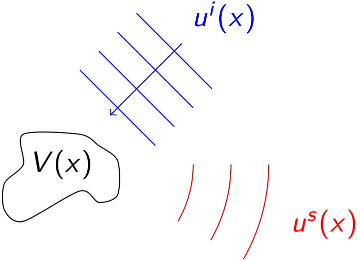
Scattering theory

Fixed frequency scattering



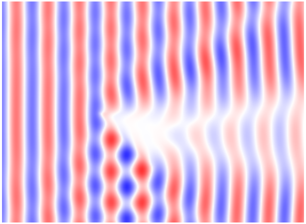
The total wave u satisfies

$$\big(\Delta + k^2(1+V)\big)u = 0,$$

V models a perturbation of the background,

$$u = \underbrace{u^i(x)}_{\text{incident wave}} + \underbrace{u^s(x)}_{\text{scattered wave}}$$

Scattering theory



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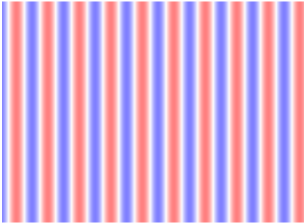

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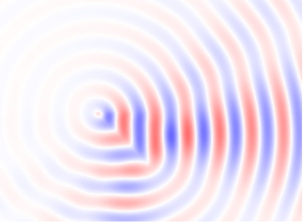
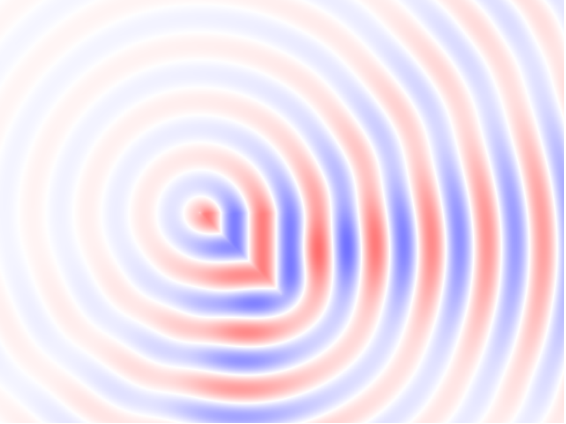


$$u = u^i + u^s$$

Fixed frequency scattering theory: measurements



Measurement: A_{u^i} is the far-field pattern of the scattered wave

$$u^s(x) = \frac{e^{ik|x|}}{|x|^{(n-1)/2}} A_{u^i}\left(\frac{x}{|x|}\right) + \mathcal{O}\left(\frac{1}{|x|^{n/2}}\right)$$

Different inverse scattering problems

Given the far-field map $u^i \mapsto A_{u^i}$, recover the scattering potential V or its support Ω.

Solved when

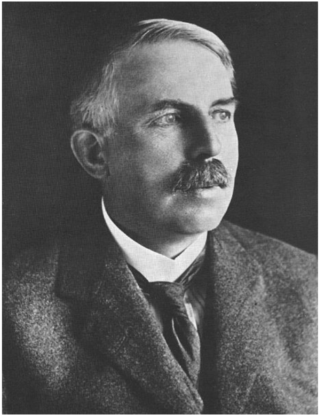
- full far-field map given for all large frequencies (Saito 1984)
- full far-field map given for a single frequency
  - Sylvester–Uhlmann 1987: 3D Calderón problem
  - R. Novikov 1988: 3D scattering
  - Bukhgeim 2007: 2D scattering
- $+$ countless other variations

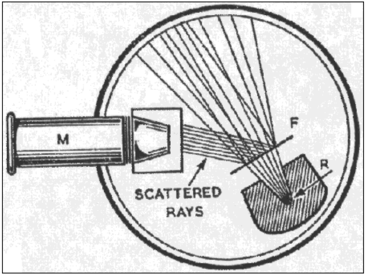
My focus is on single measurement: A_{u^i} given only for a single u^i.

Schiffer's problem: can a single measurement determine Ω?

Why one measurement only?

Example: Lord Rutherford's gold-foil experiment

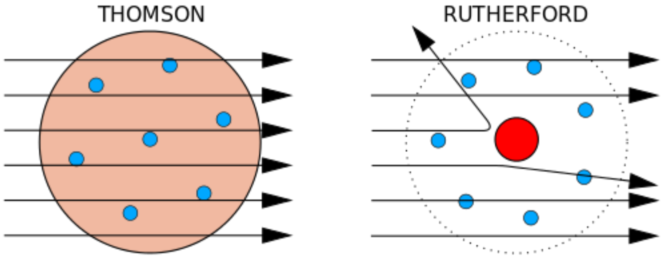




Single incident wave

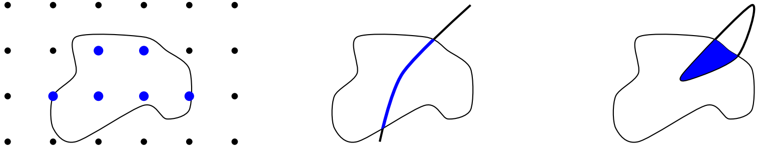
Scattering theory

Rutherford experiment's conclusions



measurement + a-priori information = conclusion

Sampling methods



- 96 Colton – Kirsh: linear sampling method (points)
- 98 Ikehata: probing method (curve)
- . . . Luke, Potthast, Sylvester, Kusiak: range test, no response test (sets)

Factorization method

Most sampling methods gave only sufficient conditions for $x \in \operatorname{supp} V$. Kirsch 90's, Grinberg 00's: factorization method. Gives necessary and sufficient conditions.

Idea:

$$u^i(x) = \int_{\mathbb{S}^{n-1}} e^{ik\theta\cdot x} g(\theta) d\sigma(\theta), \qquad g \in L^2(\mathbb{S}^{n-1})$$

$$u^s(x) = \frac{e^{ik|x|}}{|x|^{(n-1)/2}} A_g\left(\frac{x}{|x|}\right) + \mathcal{O}\left(\frac{1}{|x|^{n/2}}\right)$$

the far-field operator

$$F : L^2(\mathbb{S}^{n-1}) \to L^2(\mathbb{S}^{n-1}), \qquad Fg = A_g$$

is factored as

$$F = G\, T\, G^*$$

G compact, T isomorphism. The range of G can be characterized and gives information about $\operatorname{supp} V$.

No scattering implies k^2 ITE

Let u^i be the incident wave and assume a zero far-field: $A_{u^i} = 0$.

Rellich's lemma and unique continuation imply $u^s(x) = 0$ for $x \in \Omega = \mathbb{R}^n \setminus \operatorname{supp} V$.

$$\begin{aligned} (\Delta + k^2)u^i &= 0, && \Omega \\ (\Delta + k^2(1+V))(u^i + u^s) &= 0, && \Omega \\ u^s &\in H_0^2(\Omega), \end{aligned}$$

so $v = u^i$ and $u = u^i + u^s$ solve the interior transmission problem.

Fundamental research into ITE

- 86', 88' **Kirsch, Colton–Monk**: ITE problem posed
- 89', 91' **Colton–Kirsch–Päivärinta, Rynne–Sleeman**: discreteness of ITE
- 91'–08' NOTHING...
- 07', 09' **Cakoni–Colton–Monk, Cakoni–Colton–Haddar**: qualitative information about V from ITE's
- 08' **Päivärinta–Sylvester**: existence for general scatterers
- 10' **Cakoni–Gintides–Haddar**: infinitely many ITE's
- 10' **Cakoni–Colton–Haddar**: ITE's can be deduced from far-field data
- 11' **Hitrik–Krupchyk–Ola–Päivärinta**: bounds on location of complex ITE's
- 10'+: EXPLOSION OF INTEREST
- ∼2016: interest started shifting to "Steklov eigenvalues" http://www.maths.dur.ac.uk/lms/104/talks/1092monk.pdf

Interior transmission eigenvalues VS sampling methods

Recall: $A_{u^i} = 0, \quad u^i \neq 0 \quad \Longrightarrow \quad k^2$ ITE

Sampling method users avoid ITE's. They rely on the far-field map being injective.

Are they too careful?

- Colton–Monk 88: $\operatorname{supp} V$ compact, V radial, k^2 ITE $\Longrightarrow \exists u^i \neq 0,\ A_{u^i} = 0$
- Regge, Newton, Sabatier, Grinevich, Manakov, Novikov 50's – 90's: radial potentials transparent at a fixed k^2 i.e. $\Longrightarrow A_{u^i} = 0\ \forall u^i$

What if the measurement gives nothing?

It is very unfortunate if the far-field map is not injective. Most scattering potentials do have interior transmission eigenvalues. These exist when the map is non-injective. So it looks like the situation is unfortunate?

Theorem (B.–Päivärinta–Sylvester CMP 2014)

The potential $V = \chi_{[0,\infty[^n}\varphi$, $\varphi(0) \neq 0$ always scatters.

For any incident wave $u^i \neq 0$ and wavenumber $k > 0$ we have $A_{u^i} \neq 0$. The far-field map is injective despite there being transmission eigenvalues!

However, if k is a transmission eigenvalue A_{u^i} can become arbitrarily small with $\|u^i\| \geq 1$.

Proof sketch

Rellich's theorem and unique continuation imply $u = u^i$ in $\Omega^\complement$ so

$$k^2 \int V u^i u_0 dx = -\int_\Omega u_0 (\Delta + k^2(1+V))(u - u^i) dx = 0$$

if $(\Delta + k^2(1+V))u_0 = 0$ in Ω.
In simple case

$$\begin{aligned} u^i(x) &= u^i(0) + u^i_r(x) \\ u_0(x) &= e^{\rho\cdot x}(1+\psi(x)) \\ V(x) &= \chi_{[0,\infty[^n}(x)(\varphi(0) + \varphi_r(x)) \end{aligned}$$

Hölder estimates give

$$C \left|\varphi(0) u^i(0)\right| |\rho|^{-n} \leq \left|\varphi(0) u^i(0) \int_{[0,\infty[^n} e^{\rho\cdot x} dx\right| \leq C |\rho|^{-n-\delta}$$

if $\|\psi\|_p \leq C |\rho|^{-n/p-\varepsilon}$.

Some follow-up corner scattering results by others

- Päivärinta–Salo–Vesalainen 2017: 2D any angle, 3D almost any spherical cone
- Hu–Salo–Vesalainen 2016: smoothness reduction, new arguments, polygonal scatterer probing
- Elschner–Hu 2015, 2018: 3D any domain having two faces meet at an angle, and also curved edges
- Liu–Xiao 2017: electromagnetic waves
- ...
- free boundary methods:
  - Cakoni–Vogelius 2021: border singularities
  - Salo–Shahgholian 2021: analytic boundary non-scattering
  - ...

Injectivity of the Schiffer's problem for polyhedra

Theorem (Hu–Salo–Vesalainen, Elschner–Hu)

Let P, P' be convex polyhedra and $V = \chi_P \varphi$, $V' = \chi_{P'} \varphi'$ for admissible functions φ, φ'. Then

$$P \neq P' \quad \Longrightarrow \quad A_{u^i} \neq A'_{u^i} \quad \forall u^i \neq 0$$

Any single incident wave determines P in the class of polyhedral penetrable scatterers.

Ikehata's enclosure method (1999) gives roughly the same!

Stability of polygonal scatterer probing

Non-vanishing total wave

Theorem (B.–Liu 2021)

Let u^i be an incident wave and let $V = \chi_P \varphi$, $V' = \chi_{P'} \varphi'$ be admissible with $|u|, |u'| \neq 0$. Then

$$d_H(P, P') \leq C\left(\ln \ln \left\|A_{u^i} - A'_{u^i}\right\|_2^{-1}\right)^{-\eta}$$

for some $\eta > 0$.

Note 1: stability is still unknown without assuming $|u|, |u'| \neq 0$.
Note 2: is this the optimal stability??

Lower bound for far-field pattern

Arbitrary Herglotz wave

Theorem (B.–Liu 2017)

Let u^i be a normalized Herglotz wave,

$$u^i(x) = \int_{\mathbb{S}^{n-1}} e^{ik\theta\cdot x} g(\theta) d\sigma(\theta), \qquad \|g\|_{L^2(\mathbb{S}^{n-1})} = 1,$$

and let $V = \chi_P \varphi$ be admissible with x_c a corner of P. Then

$$\|A_{u^i}\|_{L^2(\mathbb{S}^{n-1})} \geq C_{\|P_N\|,V} > 0$$

where

$$u^i(x_c + r\theta) = r^N P_N(\theta) + \mathcal{O}(r^{N+1}),$$
$$\|P_N\| = \int_{\mathbb{S}^{n-1}} |P_N(\theta)|\, d\sigma(\theta) > 0.$$

Mistake?

F. Cakoni: "Incident waves that approximate transmission eigenfunctions produce arbitrarily small far-field patterns."

From apparent contradiction to inspiration

Theorem (B.–Liu 2017)

Let the potential $V = \chi_\Omega \varphi$ be admissible. Let $v, w \neq 0$ be transmission eigenfunctions:

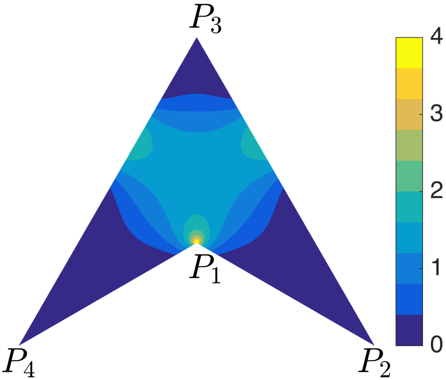
$$\begin{aligned} (\Delta + k^2)v &= 0, \qquad \Omega \\ (\Delta + k^2(1+V))w &= 0, \qquad \Omega \\ w - v &\in H_0^2(\Omega). \end{aligned}$$

Under C^α-smoothness of v near a convex corner x_c we have

$$v(x_c) = w(x_c) = 0.$$

Transmission eigenfunction localization

B.–Li–Liu–Wang 2017



Piecewise constant determination

Injectivity of piecewise constant potential probing:

Theorem (B., Liu, 2020)

Let Σ_j, $j = 1, 2, \ldots$ be bounded convex polyhedra in an admissible geometric arrangement (think pixels/voxels) and $V = \sum_j V_j \chi_{\Sigma_j}$, $V' = \sum_j V_j' \chi_{\Sigma_j}$ for constants $V_j, V_j' \in \mathbb{C}$. Then

$$V \neq V' \quad \Longrightarrow \quad A_{u^i} \neq A'_{u^i} \quad \forall u^i(x) = e^{ik\theta\cdot x}$$

if $k > 0$ small or $|u| + |u'| \neq 0$ at each vertex.

A single incident plane wave determines V in the class of discretized penetrable scatterers if the grid is unknown but same for both V and V'.

Proof sketch

Integration by parts

$$k^2 \int_\Omega (V - V')u' u_0 dx = \int_{\partial\Omega} \left((u - u')\partial_\nu u_0 - u_0 \partial_\nu (u - u')\right) dx$$

if $(\Delta + k^2(1 + V))u_0 = 0$ in Ω.
Simple case: $\Omega = B(0, \varepsilon) \cap \Sigma_j$ with $\Sigma_j =]0, 1[^n$

$$\begin{aligned} u'(x) &= u'(0) + u'_r(x) && u' \in H^2 \hookrightarrow C^{1/2} \\ u_0(x) &= e^{\rho\cdot x}(1 + \psi(x)) && \text{CGO} \\ (V - V')(x) &= V_j - V'_j && \text{piecewise constant} \end{aligned}$$

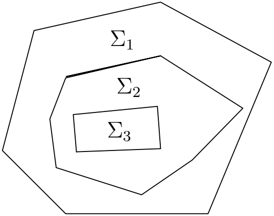
Hölder estimates give

$$C \left|(V_j - V'_j)u'(0)\right| |\rho|^{-n} = \left|(V_j - V'_j)u'(0) \int_{\mathbb{R}^n_+} e^{\rho\cdot x} dx\right| \leq C |\rho|^{-n-\delta}$$

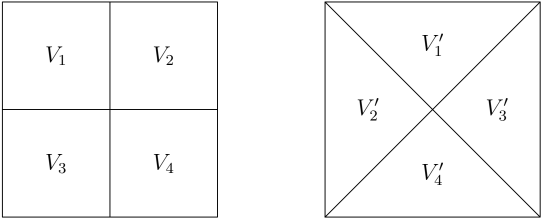
if $\|\psi\|_p \leq C |\rho|^{-n/p-\varepsilon}$.

Generalizations and limitations

- unique determination of corner location and value
- if Σ_j might be different for V, V': both $(\Sigma_j)_{j=1}^{\infty}$ and $V = \sum_j V_j \chi_{\Sigma_j}$ uniquely determined by a single measurement if geometry known to be nested



- method cannot yet be shown to distinguish between



Always scattering

High curvature case

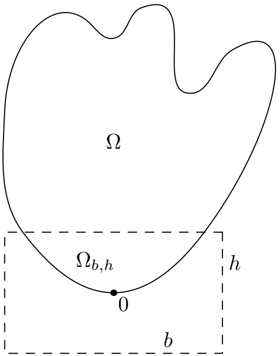
Ω bounded domain, $0 \in \partial\Omega$ admissible K-curvature point.

Theorem (B.–Liu, 2021)

If $f = \chi_\Omega \varphi$, $\varphi \in C^\alpha(\mathbb{R}^n)$ *and*

$$|\varphi(0)| \geq \mathcal{C}(\ln K)^{(n+3)/2} K^{-\delta}$$

then $u_\infty \neq 0$ *for* $(\Delta + k^2)u = f$.



Non-scattering

Technically simpler: inverse source problem

$$(\Delta + k^2)u = f, \qquad \lim_{r\to\infty} (\partial_r - ikr)\, u = 0$$

Can one have $f \neq 0$ but $u_\infty = 0$?
Recall:

$$u_\infty(\theta) = c_{k,n}\hat{f}(k\theta).$$

I.e. can a compactly supported function have Fourier transform vanishing on a sphere?
Yes: let

$$f(x) = \begin{cases} 1, & |x| < r_0 \\ 0, & |x| \geq r_0 \end{cases}$$

where $r_0 > 0$. Then

$$u_\infty(\theta) = c_{k,n}\hat{f}(k\theta) = c'_{k,n}J_{n/2}(kr_0) = 0$$

if kr_0 is a zero of the Bessel function of order $n/2$.

Always scattering

Smallness 1/2

A small uniform ball always scatters!

Also: any source with small shape always scatters!

Theorem (B.–Liu, 2021)

Let $n \geq 2$, $R_m, k \in \mathbb{R}_+$, $0 \leq \alpha \leq 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain of diameter at most R_m and whose complement is connected. Let Ω_c be a component of Ω. The source $f = \chi_\Omega \varphi$ radiates a non-zero far-field pattern at wavenumber k if

$$\big(\operatorname{diam}(\Omega_c)\big)^\alpha \leq C \frac{\sup_{\partial\Omega_c} |\varphi|}{\|\varphi\|_{C^\alpha(\overline{\Omega}_c)}},$$

for some $C = C(k, R_m, n) > 0$.

Always scattering

Smallness 2/2: Proof

Suppose $(\Delta + k^2)u = \chi_\Omega \varphi$ and $u_\infty = 0$. Then $u_{|\Omega^\complement} = 0$, so $u_{|\Omega_c} \in H_0^2(\Omega_c)$ and $(\Delta + k^2)u = \varphi$ in Ω_c.
Set $g = \varphi - k^2 u$. Elliptic regularity implies $g \in C^\alpha(\overline{\Omega}_c)$ with $\|g\|_\alpha \leq C(n, k, R_m)\|\varphi\|_\alpha$. Moreover $g = \Delta u$ and so

$$\int_{\Omega_c} g(x)dx = \int_{\Omega_c} 1 \cdot \Delta u dx = 0$$

because $u = \partial_\nu u = 0$ in $\partial\Omega_c$. Let $p \in \partial\Omega_c$. Then

$$\varphi(p)m(\Omega_c) = g(p)m(\Omega_c) = -\int_{\Omega_c} \big(g(x) - g(p)\big)dx$$

Hence

$$|\varphi(p)|\, m(\Omega_c) \leq \|g\|_\alpha \int_{\Omega_c} |x - p|^\alpha\, dx \leq \|g\|_\alpha\, m(\Omega_c)\big(\operatorname{diam}(\Omega_c)\big)^\alpha.$$

Inverse source problem, Schiffer's problem

$$(\Delta + k^2)u = f = \chi_\Omega \varphi, \qquad \lim_{r\to\infty}(\partial_r - ikr)u = 0$$

Can $u_\infty(\theta) = c\hat{f}(k\theta)$ determine Ω when k is fixed?

Unique determination:

- $u_\infty = u'_\infty \Longrightarrow \Omega = \Omega'$ for convex polyhedral shapes (corner scattering). Assuming non-vanishing total waves, also for elasticity (B.–Lin 2018), electromagnetism (B.–Liu–Xiao 2021),
- $u_\infty = u'_\infty \Longrightarrow \Omega \approx \Omega'$ for convex polyhedral shapes whose corners have been smoothened to admissible K-curvature points (high curvature scattering, B.–Liu 2021),
- $u_\infty = u'_\infty \Longrightarrow \Omega \approx \Omega'$ for well-separated collections of small scatterers (small source scattering, B.–Liu 2021).

Non-spherical cones

Potential scattering

Let C be any cone whose cross-section K is star-shaped and $\chi_K \in H^\tau(\mathbb{R}^2)$ for some $\tau > 1/2$.

Theorem (B.–Pohjola 2022)

For any $\delta > 0$ there is a cone C_δ such that $d_H(C_\delta, C) < \delta$ and with the following property: potentials of the form

$$V = \chi_{C_\delta}\varphi$$

where φ is smooth enough (roughly $C^{1/4}$) and non-zero at the vertex always scatter.

Non-spherical cones

Source scattering (easier)

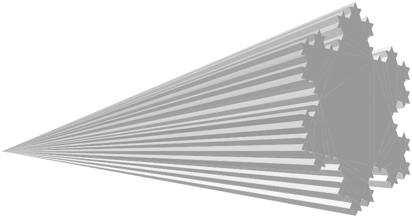
Theorem (B.–Pohjola 2022)

A source $f = \chi_C \varphi$ for $(\Delta + k^2)u = f$ scatters for any $k > 0$ when φ is smooth enough and non-zero at the vertex of the cone C when

$$\int_{\mathbb{S}^2 \cap C} Y_2^m dS \neq 0$$

for $m \in \{-2, -1, 0, +1, +2\}$ and Y_2^m is the spherical harmonic of degree 2. This is true if C fits into a thin enough spherical cone.

"Thin enough" means $\cos\theta \leq 1/\sqrt{3}$. The magic angle is $\approx 54.74^\circ$.



Scattering screens

A flat screen $\Omega = \Omega_0 \times \{0\}$ with $\Omega_0 \subset \mathbb{R}^2$ simply connected, bounded and smooth. Scattering from such a screen:

$$\begin{aligned} (\Delta + k^2)u^s &= 0, && \mathbb{R}^3 \setminus \overline{\Omega}, \\ u^i + u^s &= 0, && \Omega, \\ r(\partial_r - ik)u^s &\to 0, && r = |x| \to \infty. \end{aligned}$$

Let Ω, Ω' be flat screens, $k > 0$, u^i an arbitary incident wave, and $u^s, u^{s\prime}$ corresponding scattered waves.

Theorem (B.–Päivärinta–Sadique 2020)

- *If $u^i(x_1, x_2, x_3) + u^i(x_1, x_2, -x_3) \neq 0$ for some x and $u^s_\infty = {u^{s\prime}}_\infty$ then $\Omega = \Omega'$.*
- *If $u^i(x_1, x_2, x_3) + u^i(x_1, x_2, -x_3) = 0$ for all x then $u^s_\infty = {u^{s\prime}}_\infty = 0$.*

What about the future?

New directions: free boundary methods. Will they solve the problem?

What is the problem?

What geometric features of a scatterer cause arbitrary

a) plane waves,
b) Herglotz or other waves

to give non-trivial scattering?

What guarantees vanishing far-fields?

Thank you!