Imaging water supply networks and vocal tracts

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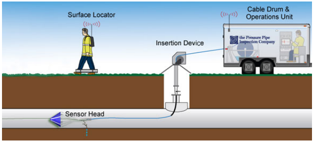
Water supply network

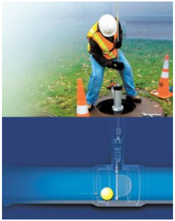
How to locate problems traditionally?

Sahara System



Replace & Rehabilitate

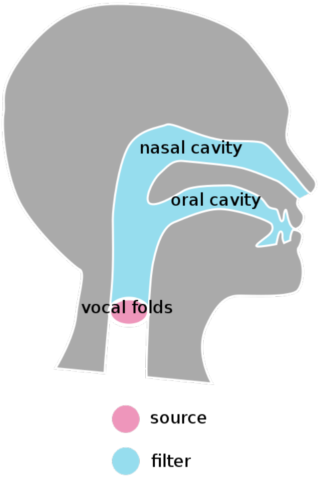
Smart Ball

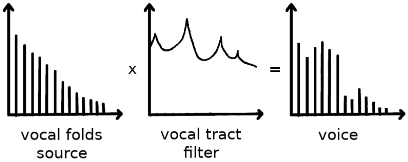


Sonar

Gas Injection

Vocal tract imaging: inverse glottal filtering



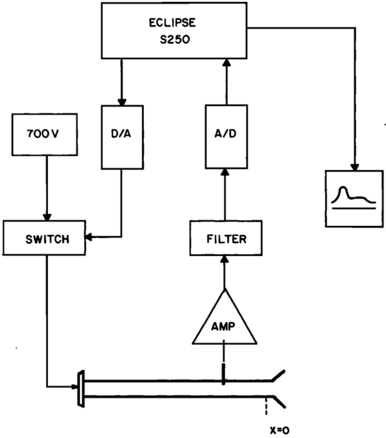


output $=$ source $*$ filter

Find cheap and convenient ways to get *useful information* about the source or the filter!

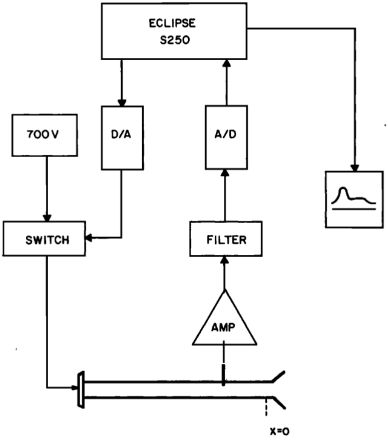
Difficulty: IGF is a *blind deconvolution*!

Vocal tract imaging: with an external sound



Sondhi, M. M., & Resnick, J. R. (1983). The inverse problem for the vocal tract: numerical methods, acoustical experiments, and speech synthesis. The Journal of the Acoustical Society of America, 73(3), 985–1002. http://dx.doi.org/10.1121/1.389024

Vocal tract imaging: with an external sound



Sondhi, M. M., & Resnick, J. R. (1983). The inverse problem for the vocal tract: numerical methods, acoustical experiments, and speech synthesis. The Journal of the Acoustical Society of America, 73(3), 985–1002. `http://dx.doi.org/10.1121/1.389024`

Alternative source: white noise, it contains all the frequencies!

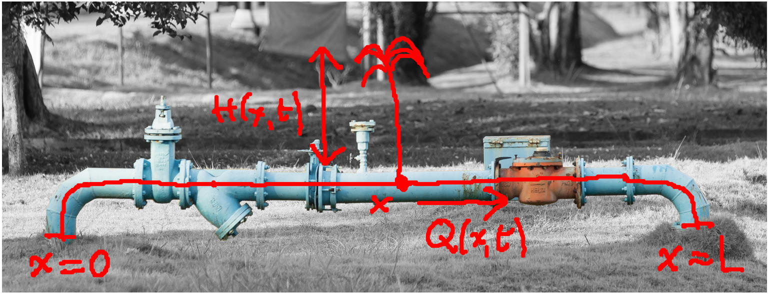
Mathematical model

Water in pipes
VS
Air in the vocal tract

Fluid in a pipe. Same model!

Direct model: single pipe

Direct model: single pipe



$$
\begin{aligned}
\partial_t H + \frac{a^2}{gA}\partial Q &= 0, && 0 < x < L, \quad t \in \mathbb{R}, \\
\partial_t Q + gA\partial H &= 0, && 0 < x < L, \quad t \in \mathbb{R}, \\
H = Q &= 0, && 0 < x < L, \quad t \leq 0.
\end{aligned}
$$

Water hammer equations.

https://www.youtube.com/watch?v=jTrhHUwDNYE

One pipe inverse problem

Measurement $\Lambda(t)$ defined by

1. assuming stable situation,
2. send a flow δ-pulse from $x = 0$,
3. measure the pressure at $x = 0$.

In other words:

$$Q(0, t) = \delta_0(t) \implies \Lambda(t) = H(0, t).$$

Inverse problem: recover $A(x)$ given measurement $\Lambda(t)$.

Integration by parts

For simplicity assume $a(x) = a_0$ constant! Assume virtual H_v, Q_v causal solutions. Then

$$-\partial Q_v = \frac{gA}{a_0^2}\partial_t H_v$$

and integrate $\int_0^\tau \int_0^{a_0\tau} \dots dxdt$ given any fixed $\tau > 0$.

$$-\int_0^\tau \int_0^{a_0\tau} \partial Q_v(x,t)dxdt = \int_0^\tau \int_0^{a_0\tau} \frac{gA(x)}{a_0^2}\partial_t H_v(x,t)dxdt$$

- $H_v = Q_v = 0$ at $t = 0$
- hence $H_v(x,t) = Q_v(x,t) = 0$ when $x \geqslant a_0 t$, so

$$\int_0^\tau Q_v(0,t)dt = \int_0^{a_0\tau} \frac{gA(x)}{a_0^2} H_v(x,\tau)dx \tag{1}$$

Special solutions

Given any causal solutions, for example the virtual ones H_v, Q_v, let's look at the total volume input into the system:

$$V(\tau) := \int_0^{\tau} Q_v(0,t)dt = \int_0^{a_0\tau} \frac{gA(x)}{a_0^2} H_v(x,\tau)dx.$$

If H_v is such that

$$H_v(x,\tau) = \begin{cases} 1, & x < a_0\tau \\ 0, & x \geqslant a_0\tau \end{cases} \quad \text{at } t = \tau,$$

then

$$A(x) = \frac{a_0}{g}\frac{\partial V}{\partial \tau}\left(\frac{x}{a_0}\right)$$

After the facts, new problem statement

Unknown: $A(x)$. Measurement:

$$Q(0, t) = \delta_0(t), \qquad H(0, t) = \Lambda(t).$$

Given any τ, find causal solutions H_v, Q_v such that at time $t = \tau$

$$H_v(x, \tau) = \begin{cases} 1, & x < a_0\tau \\ 0, & x \geqslant a_0\tau \end{cases}$$

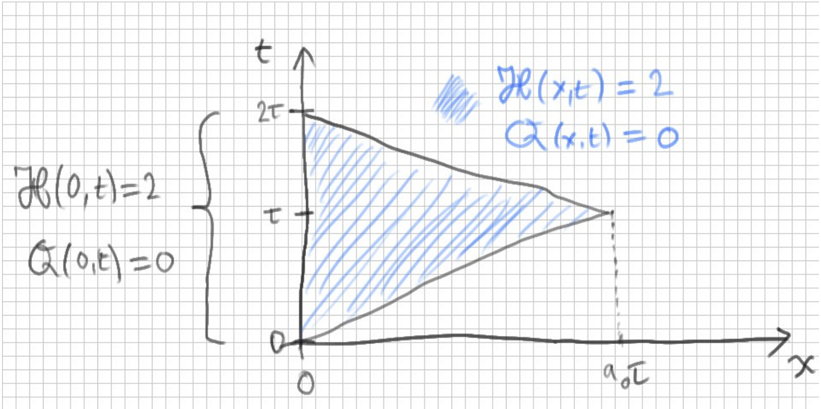
The corresponding Q_v will be used to calculate $A(x)$.

Unique continuation

If $\mathscr{H}, \mathscr{Q}$ satisfy the equations on $0 < x < L$, $0 < t < 2\tau$ (but are not necessarily causal), and

$$\mathscr{H}(0,t) = 2, \quad \mathscr{Q}(0,t) = 0, \qquad 0 < t < 2\tau,$$

then $\mathscr{H}(x,t) = 2$, $\mathscr{Q}(x,t) = 0$ in $x < a_0(\tau - |\tau - t|)$.



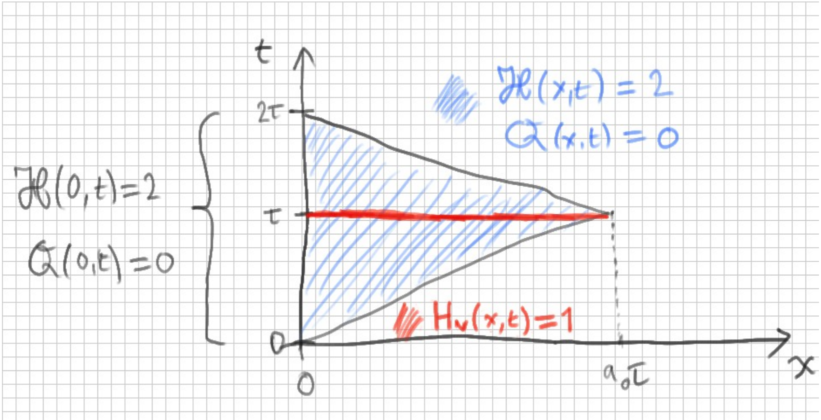
Unique continuation to causal solutions

If $\mathscr{H}, \mathscr{Q}$ as previously and H_v, Q_v causal, and

$$\mathscr{H}(x,t) = H_v(x,t) + H_v(x, 2\tau - t), \quad \mathscr{Q}(x,t) = Q_v(x,t) - Q_v(x, 2\tau - t)$$

$$\implies H_v(x,\tau) = \tfrac{1}{2}\mathscr{H}(x,\tau) = \begin{cases} 1, & x < a_0\tau \\ 0, & x \geqslant a_0\tau \end{cases}$$

Then $A(a_0\tau) = a_0 g^{-1}\partial_\tau \int_0^\tau Q_v(0,t)dt$. (Note: Q_v depends on τ!!)



Next?

How to find the suitable H_v, Q_v?

Integral equation from requirements of $\mathscr{H}, \mathscr{Q}$

Measurement:

$$Q(0,t) = \delta_0(t), \quad H(0,t) = \frac{a_0}{gA(0)}(\delta_0(t) + \mathbf{h(t)})$$

Let H_v, Q_v be causal solutions such that $\mathscr{H}(0,t) = 2$, $\mathscr{Q}(0,t) = 0$ on $0 < t < 2\tau$. Then rewriting these eq's in term of Q_v gives:

$$Q_v(0,t) + \frac{1}{2}\int_0^{2\tau} Q_v(0,s)\mathbf{h(|s-t|)}ds = \frac{gA(0)}{a_0}, \qquad 0 < t < 2\tau.$$

Conversely, if Q_v solves the above and H_v is the corresponding pressure head, then

$$H_v(x,\tau) = \begin{cases} 1, & x < a_0\tau \\ 0, & x \geqslant a_0\tau \end{cases} \qquad \text{at } t = \tau,$$

so $A(a_0\tau) = a_0 g^{-1}\partial_\tau \int_0^\tau Q_v(0,t)dt$

Algorithm

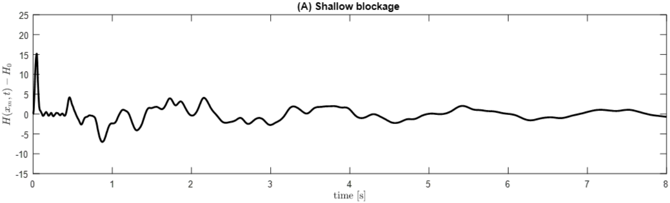
1. Input $Q(0,t) = \delta_0(t)$ and for $t < 2T = 2L/a_0$ measure
$$H(0,t) = \frac{a_0}{gA(0)}(\delta_0(t) + h(t))$$
2. For $0 < \tau < T$ solve for the boundary value of the virtual solution Q_v
$$Q_v(0,t) + \frac{1}{2}\int_0^{2\tau} Q_v(0,s)h(|s-t|)ds = \frac{gA(0)}{a_0}, \quad 0 < t < 2\tau$$
3. Set
$$V(\tau) = \int_0^{\tau} Q_v(0,t)dt \qquad \left(= \int_0^{a_0\tau} \frac{gA(x)}{a_0^2}dx \right)$$
4. Repeat 2–3 (on the computer) for many τ to get a good approximation of V
5. Given $x < L$ the area can be found by
$$A(x) = \frac{a_0}{g}\left(\frac{\partial}{\partial\tau}V(\tau)\right)_{\tau = x/a_0}$$

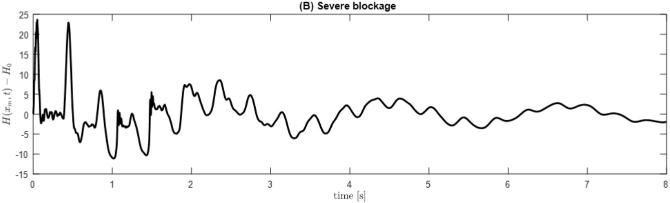
Laboratory experiment: setup

Measurement set up by Silvia Meniconi and Bruno Brunone's group.

Laboratory experiment: impulse-response function

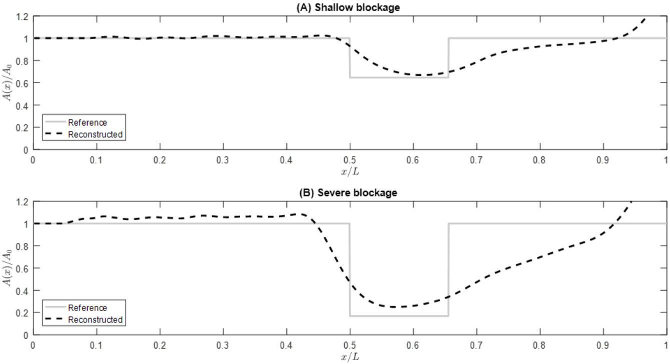
Measurement set up by Silvia Meniconi and Bruno Brunone's group.



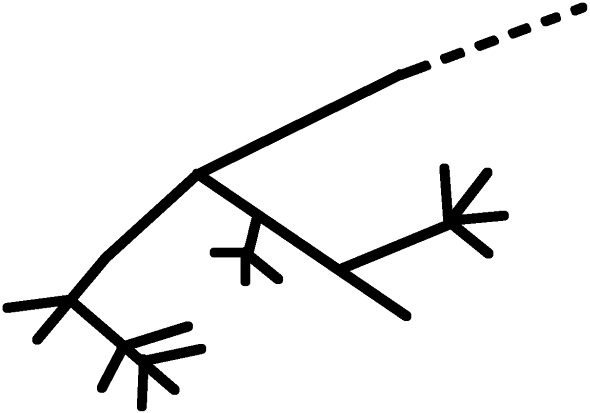


Reconstruction from measured and processed data

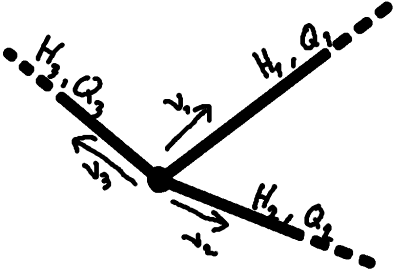
Measurement set up by Silvia Meniconi and Bruno Brunone's group.



Network



Junction conditions



- H is a scalar: the boundary values H_j are the same at connected pipe ends
- No sinks or sources (total water flowing into pipes sum to zero):

$$\sum_j \nu_j Q_j = 0$$

Main difficulty compared to a segment?

The sets where we can have $H_v(x, \tau) = 1$.

In which Ω can we force $H_v(x,\tau) = 1$?

Control theory suggests that there are boundary values such that

$$H_v(x,\tau) = \begin{cases} 1, & x \in \Omega \\ 0, & x \notin \Omega \end{cases}$$

given any measurable set $\Omega \subset \mathbb{G}$ when $\mathbb{G}$ is a tree and τ large.

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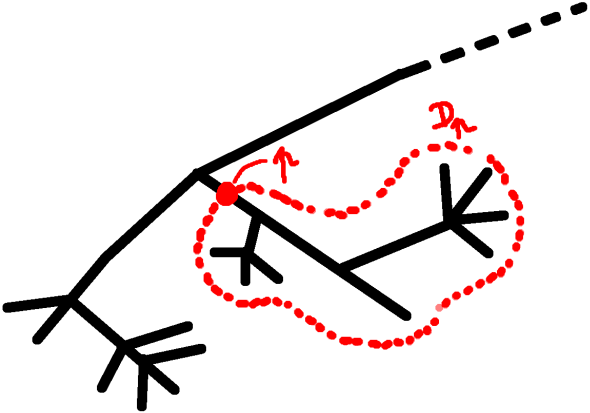
HOWEVER

Can we solve for these boundary values? Is it computationally efficient? Is it even possible?

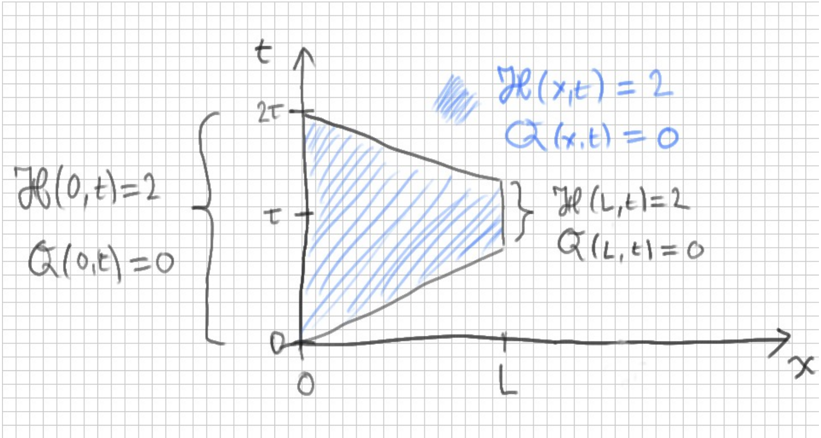
Admissible domains

This works:



But needs a matrix of measurements!

Inductive unique continuation



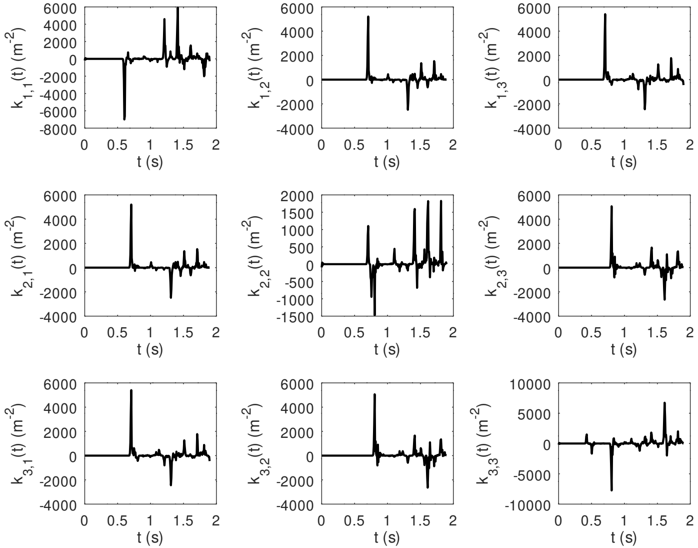
$+$ junction conditions
$=$ propagate $\mathscr{H} = 2$, $\mathscr{Q} = 0$!!

The same logic as before works, but all the equations become more complicated.

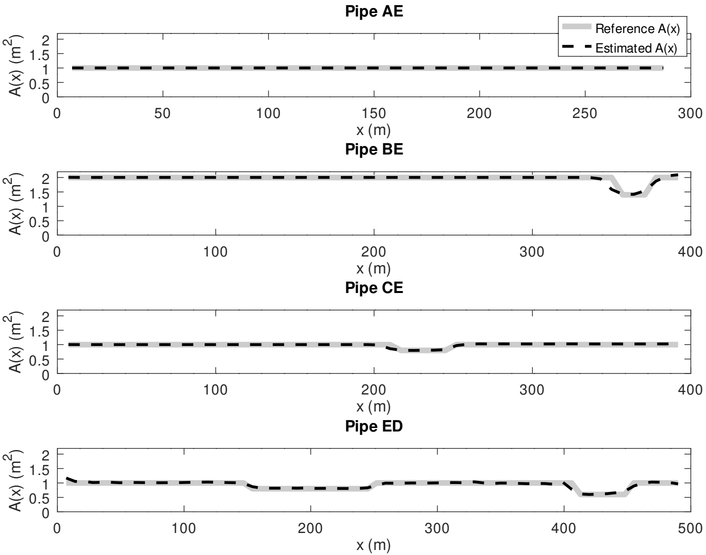
Numerical experiment: setup

Numerical experiment: impulse-response matrix function

$h_{ij}(t) = A(x_j)g/a_0 \cdot k_{ij}(t)$



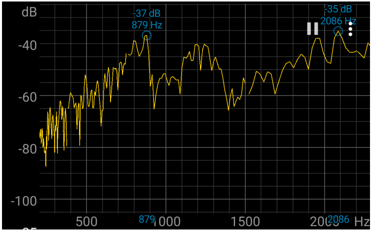
Reconstruction from measured data using regularization

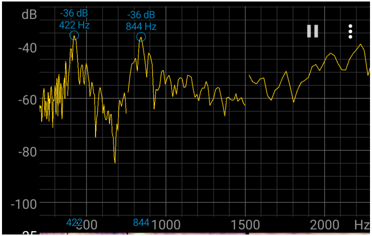


Ongoing work

With Lauri Oksanen (Univ. Helsinki) and Tapio Helin (LUT)

Instead of $Q(0,t) = \delta_0(t)$ we have white noise $Q(0,t) = W(t)$.





Kiitos!