Detecting blockages in water supply networks using boundary control

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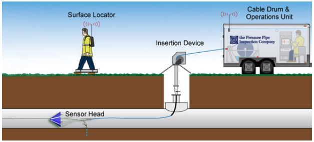
Water supply network

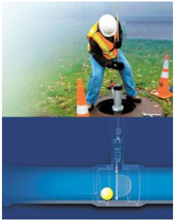
How to locate problems traditionally?

Sahara System



Replace & Rehabilitate

Smart Ball

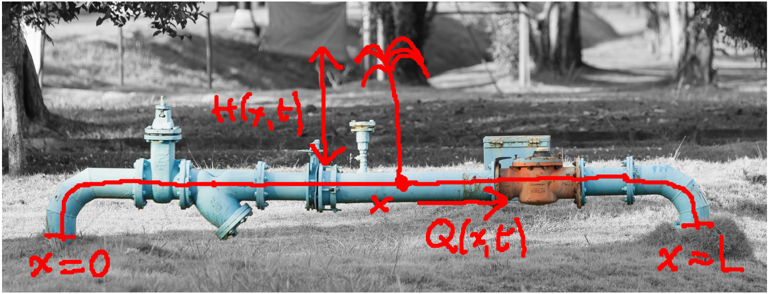


Sonar

Gas Injection

Direct model: single pipe

Direct model: single pipe



$$
\begin{aligned}
\partial_t H + \frac{a^2}{gA}\partial Q &= 0, & 0 < x < L, \quad & t \in \mathbb{R}, \\
\partial_t Q + gA\partial H &= 0, & 0 < x < L, \quad & t \in \mathbb{R}, \\
H = Q &= 0, & 0 < x < L, \quad & t \leq 0.
\end{aligned}
$$

One pipe inverse problem

Measurement $\Lambda(t)$ defined by assuming H, Q satisfy

$$
\begin{aligned}
\partial_t H + \frac{a^2}{gA}\partial Q = 0, &\qquad 0 < x < L, \quad t \in \mathbb{R}, \\
\partial_t Q + gA\partial H = 0, &\qquad 0 < x < L, \quad t \in \mathbb{R}, \\
H = Q = 0, &\qquad 0 < x < L, \quad t \leq 0
\end{aligned}
$$

Boundary conditions:

unknown at $x = L$

unit impulse discharge $Q(0, t) = \delta_0(t)$ at $x = 0$

Then $\Lambda(t) = H(0, t)$.

Recover: $A(x)$ (or $a(x)$, or $A(x)/a(x)$)

Integration by parts

For simplicity assume $a(x) = a_0$ constant! Assume H, Q causal. Then

$$\partial_t H + \frac{a_0^2}{gA}\partial Q = 0$$

and integrate $\int_0^\tau \int_0^{a_0\tau} \dots dxdt$ given any fixed $\tau > 0$.

$$-\int_0^\tau \int_0^{a_0\tau} \partial Q(x,t)dxdt = \int_0^\tau \int_0^{a_0\tau} \frac{gA(x)}{a_0^2}\partial_t H(x,t)dxdt$$

- $H = Q = 0$ at $t = 0$
- hence $H(x,t) = Q(x,t) = 0$ when $x \geqslant a_0 t$, so

$$\int_0^\tau Q(0,t)dt = \int_0^{a_0\tau} \frac{gA(x)}{a_0^2} H(x,\tau)dx \qquad (1)$$

Special solutions

Given causal solutions H, Q we know from boundary measurements the value of

$$V(\tau) := \int_0^{\tau} Q(0,t)dt = \int_0^{a_0\tau} \frac{gA(x)}{a_0^2} H(x,\tau)dx.$$

If H such that

$$H(x,\tau) = \begin{cases} 1, & x < a_0\tau \\ 0, & x \geqslant a_0\tau \end{cases} \quad \text{at } t = \tau,$$

then

$$A(x) = \frac{a_0}{g}(\partial V)\left(\frac{x}{a_0}\right)$$

Unique continuation

If $\mathscr{H}, \mathscr{Q}$ satisfy the equations on $0 < x < L$, $0 < t < 2\tau$ (but are not necessarily causal), and

$$\mathscr{H}(0,t) = 2, \quad \mathscr{Q}(0,t) = 0, \qquad 0 < t < 2\tau,$$

then $\mathscr{H}(x,t) = 2$, $\mathscr{Q}(x,t) = 0$ in $x < a_0(\tau - |\tau - t|)$.

If

$$\mathscr{H}(x,t) = H(x,t) + H(x,2\tau - t), \quad \mathscr{Q}(x,t) = Q(x,t) - Q(x,2\tau - t)$$

where H, Q causal solutions, AND $\mathscr{H}, \mathscr{Q}$ satisfy the above, then

$$2H(x,\tau) = \mathscr{H}(x,\tau) = \begin{cases} 2, & x < a_0\tau \\ 0, & x \geqslant a_0\tau \end{cases}$$

and so $A(a_0\tau) = a_0 g^{-1} \partial_\tau \int_0^\tau Q(0,t)dt$. ($Q$ depends on τ!!)

Integral equation from requirements of $\mathscr{H}, \mathscr{Q}$

Measurement (if A constant near $x = 0$):

$$\Lambda(t) = \frac{a_0}{gA(0)}(\delta_0(t) + h(t))$$

Let H, Q be causal solutions such that $\mathscr{H}(0,t) = 2$, $\mathscr{Q}(0,t) = 0$ on $0 < t < 2\tau$. Then

$$Q(0,t) + \frac{1}{2}\int_0^{2\tau} Q(0,s)h(|s-t|)ds = \frac{gA(0)}{a_0}, \qquad 0 < t < 2\tau.$$

Conversely, if Q solves the above and H is the corresponding pressure head, then

$$H(x,\tau) = \begin{cases} 1, & x < a_0\tau \\ 0, & x \geqslant a_0\tau \end{cases} \qquad \text{at } t = \tau,$$

so $A(a_0\tau) = a_0 g^{-1}\partial_\tau \int_0^\tau Q(0,t)dt$

Algorithm (SKIP?)

1. input $Q(0,t) = \delta_0(t)$ and measure $\Lambda(t) = H(0,t)$ for $t < 2T = 2L/a_0$
2. for $0 < \tau < T$ solve

$$Q(0,t) + \frac{1}{2}\int_0^{2\tau} Q(0,s)h(|s-t|)ds = \frac{gA(0)}{a_0}, \qquad 0 < t < 2\tau$$

3. set

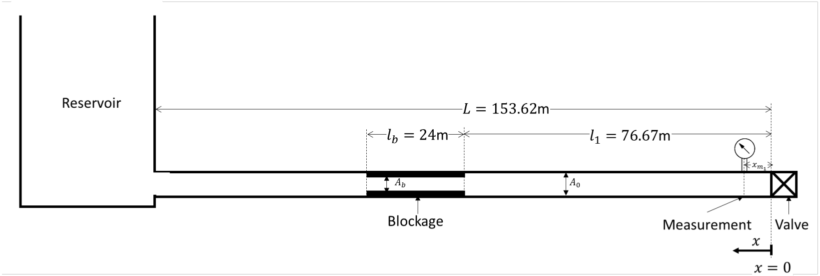
$$V(\tau) = \int_0^{\tau} Q(0,t)dt \qquad \left(= \int_0^{a_0\tau} \frac{gA(x)}{a_0^2}dx\right)$$

4. repeat 2–3 (in the computer) for many τ to get a good approximation of $V(\tau)$
5. given $x < L$ the area can be found by

$$A(x) = \frac{a_0}{g}\left(\frac{\partial}{\partial\tau}V(\tau)\right)_{\tau = x/a_0}$$

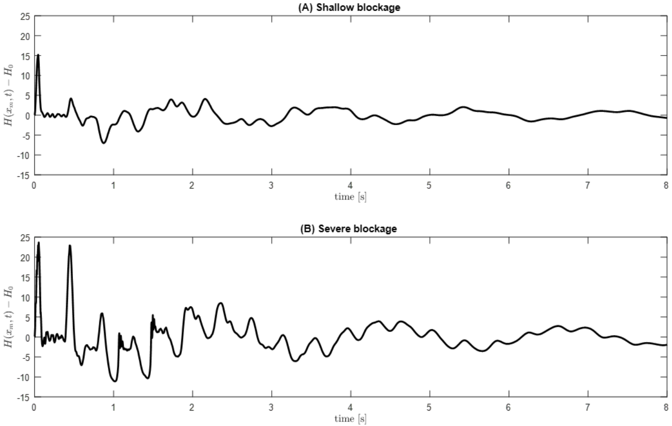
Laboratory experiment: setup

Measurement set up by Silvia Meniconi and Bruno Brunone's group.



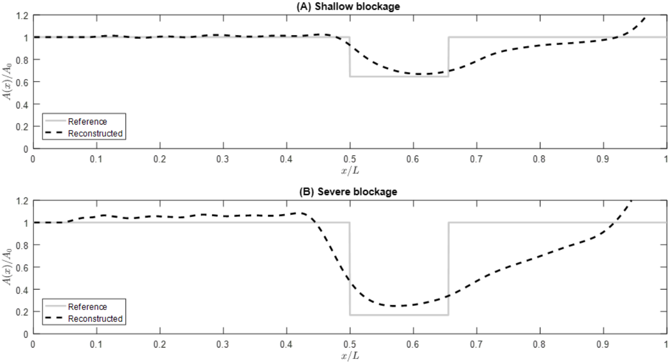
Laboratory experiment: impulse-response function

Measurement set up by Silvia Meniconi and Bruno Brunone's group.

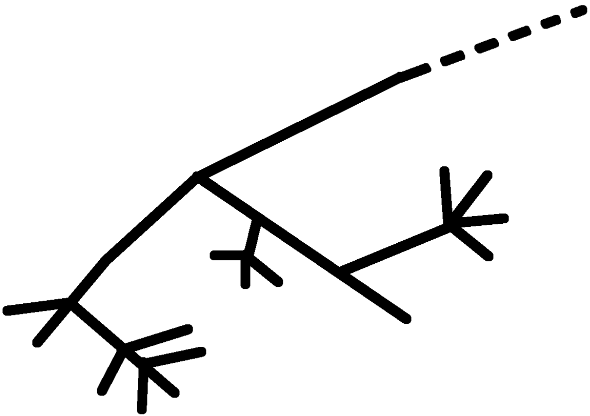


Reconstruction from measured and processed data

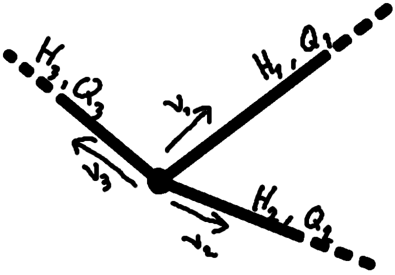
Measurement set up by Silvia Meniconi and Bruno Brunone's group.



Network



Junction conditions



- H is a scalar: the boundary values H_j are the same at connected pipe ends
- No sinks or sources (total water flowing into pipes sum to zero):

$$\sum_j \nu_j Q_j = 0$$

Integration by parts

Integrate one of the equations over a time-interval $0 < t < \tau$ and the whole network $x \in \mathbb{G}$. This gives

$$\sum_{x_j \in \partial \mathbb{G}} \int_0^{\tau} \nu(x_j) Q(x_j, t) dt = \int_{\mathbb{G}} \frac{gA(x)}{a_0^2} H(x, \tau) dx$$

if $H(x, t) = 0$ for $t \leq 0$.

Let $\Omega \subset \mathbb{G}$. If there are boundary flows $Q(x_j, t)$ so that

$$H(x, \tau) = \begin{cases} 1, & x \in \Omega, \\ 0, & x \notin \Omega, \end{cases} \quad \text{at time } t = \tau.$$

Then we have

$$V(\Omega) := \sum_{x_j \in \partial \mathbb{G}} \int_0^{\tau} \nu(x_j) Q(x_j, t) dt = \frac{g}{a_0^2} \int_{\Omega} A(x) dx.$$

In which Ω can we force $H(x, \tau) = 1$?

Control theory suggests that there are boundary values such that $H(x, \tau) = \chi_\Omega(x)$ given any measurable set $\Omega \subset \mathbb{G}$ when $\mathbb{G}$ is a tree.

In which Ω can we force $H(x, \tau) = 1$?

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HOWEVER

Can we solve for these boundary values? Is it computationally efficient? Is it even possible?

Use the one pipe method

What if we set $\mathscr{H}(x_j, t) = 2$, $\mathscr{Q}(x_j, t) = 0$ on boundary points $x_j \in \partial\mathbb{G}$ for some time-intervals $\tau - f(x_j) < t < \tau + f(x_j)$?

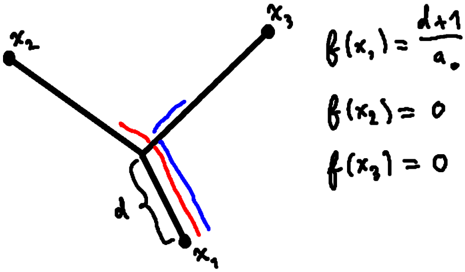
Use the one pipe method

What if we set $\mathscr{H}(x_j, t) = 2$, $\mathscr{Q}(x_j, t) = 0$ on boundary points $x_j \in \partial\mathbb{G}$ for some time-intervals $\tau - f(x_j) < t < \tau + f(x_j)$?

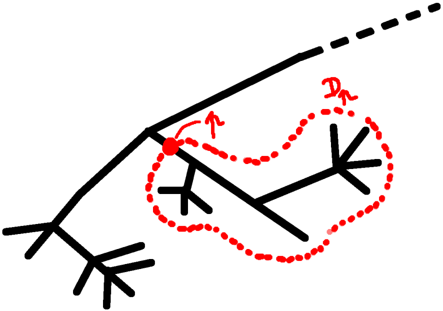
Unique continuation does NOT guarantee that $\mathscr{H}(x, \tau) = 2$ in the *domain of influence*

$$\{x \in \mathbb{G} \mid d(x, x_j) < a_0 f(x_j) \text{ for some } x_j \in \partial\mathbb{G}\}.$$

For example:



Admissible domains



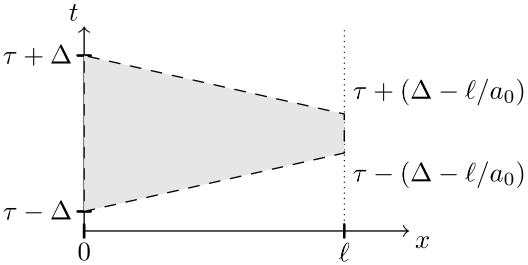
If $p \in \mathbb{G}$ and

$$f(x_j) = \begin{cases} d(x_j, p)/a_0, & x_j \in \partial D_p \cap \partial \mathbb{G}, \\ 0, & x_j \in \partial \mathbb{G} \setminus \partial D_p. \end{cases}$$

then

$$D_p = \{x \in \mathbb{G} \mid d(x, x_j) < a_0 f(x_j) \text{ for some } x_j \in \partial \mathbb{G}\}.$$

Inductive unique continuation



+ junction conditions
= propagate $\mathscr{H} = 2$, $\mathscr{Q} = 0$!!

If $\mathscr{H}, \mathscr{Q}$ satisfy the equations on $x \in \mathbb{G}$, $t \in \mathbb{R}$ (but are not necessarily causal), and

$$\mathscr{H}(x_j, t) = 2, \quad \mathscr{Q}(x_j, t) = 0, \qquad x_j \in \partial\mathbb{G}, \tau - f(x_j) < t < \tau + f(x_j),$$

then $\mathscr{H}(x, \tau) = 2$, $\mathscr{Q}(x, \tau) = 0$ in D_p at $t = \tau$.

Network integral equation

Let $Q_p(x_j, t)$ satisfy (x_0 is end to which we do not have access)

$$\frac{gA(x_j)}{a_0} = \nu(x_j)Q_p(x_j, t) + \sum_{\substack{x_i \in \partial\mathbb{G} \\ x_i \neq x_0}} \frac{\nu(x_i)}{2} \int_0^\tau Q_p(s, x_i)\big(h_{ij}(|t-s|) + h_{ij}(2\tau - t - s)\big)ds$$

when $\tau - f(x_j) < t \leq \tau$ and

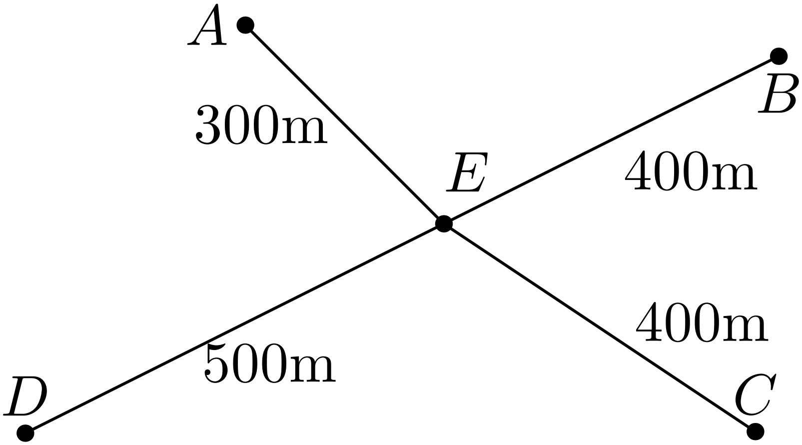
$$Q_p(x_j, t) = 0$$

when $0 \leq t \leq \tau - f(x_j)$.

Then if H, Q satisfy the network wave model with boundary flow νQ_p we have

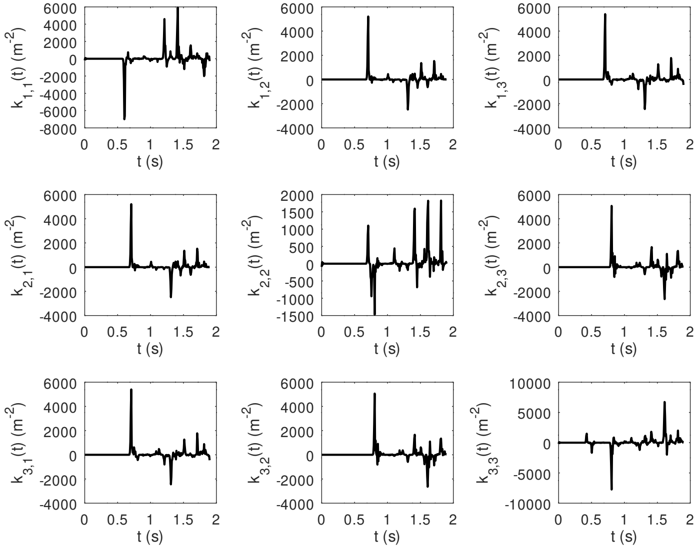
$$H(x, \tau) = \begin{cases} 1, & x \in D_p, \\ 0, & x \in \mathbb{G} \setminus D_p. \end{cases}$$

Numerical experiment: setup

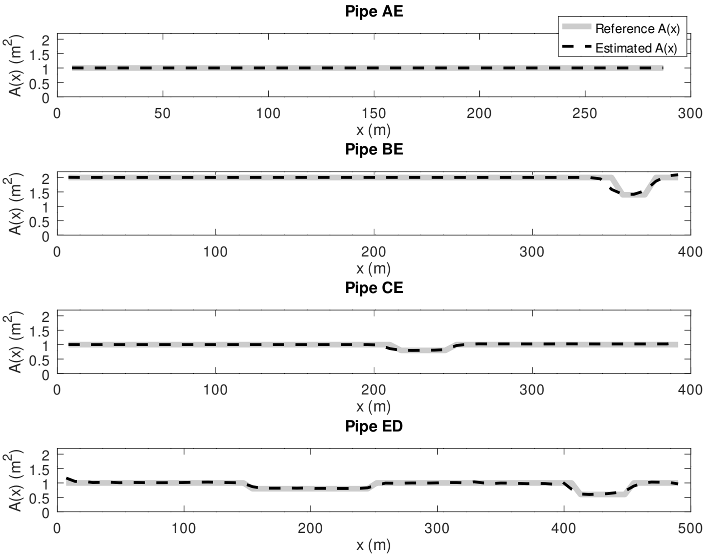


Numerical experiment: impulse-response matrix function

$h_{ij}(t) = A(x_j)g/a_0 \cdot k_{ij}(t)$



Reconstruction from measured data using regularization



Existence of solution?

If

$$\tilde{Q}(x_j, t) = \begin{cases} \nu(x_j) Q_p(t, x_j), & 0 \le t \le \tau, \\ \nu(x_j) Q_p(2\tau - t, x_j), & \tau < t \le 2\tau, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\tilde{1}(x_j, t) = \begin{cases} 1, & |t - \tau| < f(x_j), \\ 0, & |t - \tau| \ge f(x_j), \end{cases}$$

then the equations for Q_p can be written as an operator equation

$$\mathscr{K} \tilde{Q}(x_j, t) = \frac{gA(x_j)}{a_0} \tilde{1}(x_j, t).$$

Functional analysis

$$\mathscr{K}\tilde{Q}(x_j,t) = \frac{gA(x_j)}{a_0}\tilde{1}(x_j,t), \qquad \tilde{Q} \in L_0^2$$

where $\mathscr{K} : L_0^2 \to L_0^2$ are defined by

$$L_0^2 = \Big\{ \tilde{Q} \in L^2(\partial\mathbb{G} \setminus \{x_0\} \times \mathbb{R}) \,\Big|\, \tilde{Q}(x_j,t) = 0 \text{ if } |t-\tau| \geq f(x_j),\ \tilde{Q}(x_j, 2\tau - t) = \tilde{Q}(x_j,t) \Big\}.$$

and

$$\mathscr{K}\tilde{Q}(x_j,t) = \tilde{Q}(x_j,t) + \sum_{x_i \in \partial\mathbb{G}\setminus\mathbb{J}} \frac{\tilde{1}(x_j,t)}{2} \int_0^{2\tau} \tilde{Q}(x_i,s) h_{ij}(|t-s|)ds.$$

$\mathscr{K}$ is self-adjoint, Fredholm, pos. semidef. and $gA\tilde{1}/a_0 \perp \ker \mathscr{K}$!!

Kiitos!