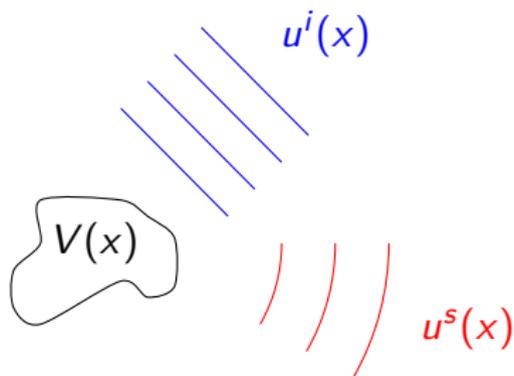# Inverse problems with one measurement

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National University of Singapore
Singapore, August 15, 2018

Scattering theory
Fixed frequency scattering
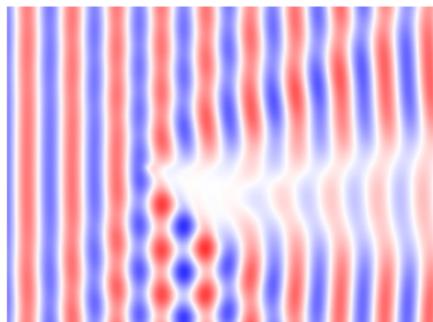

The total wave u satisfies

$$(\Delta + k^2(1 + V))u = 0,$$
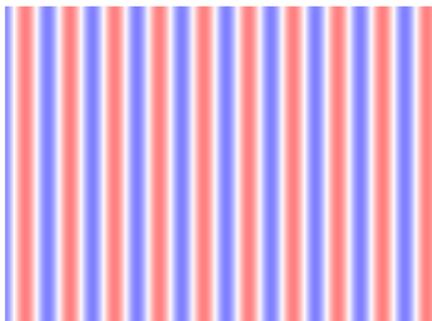
V models a perturbation of the background,

$$u = u^i(x) + u^s(x)$$

incident wave   scattered wave
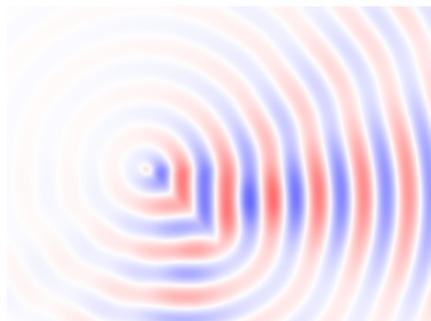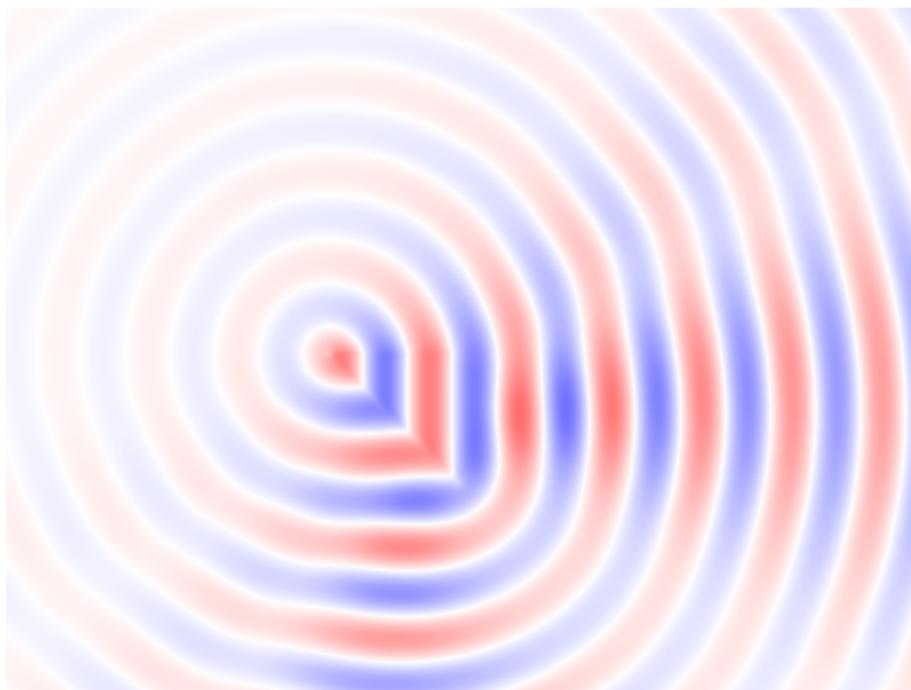
Scattering theory

=

+

Mathematical scattering theory: measurements

Measurement: A_{u^i} is the far-field pattern of the scattered wave

$$u^s(x) = \frac{e^{ik|x|}}{|x|^{(n-1)/2}} A_{u^i}\left(\frac{x}{|x|}\right) + \mathcal{O}\left(\frac{1}{|x|^{n/2}}\right)$$

Inverse problems

Given the far-field map $u^i \mapsto A_{u^i}$, recover V or its support Ω.

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Solved when

▶ full far-field map given for all large frequencies (Saito 84),

▶ full far-field map given for a single frequency
(Sylvester–Uhlmann 87 $n \geq 3$ + Bukhgeim 07 $n = 2$),

▶ + countless other variations

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Focus on *single measurement*: A_{u^i} given only for a single u^i.

Schiffer's problem: can a single measurement determine Ω?

What about in physics?
Lord Rutherford's gold-foil experiment

Single incident wave

measurement + *a-priori information* = conclusion

What if the measurement gives nothing?

If $A_{u^i} = 0$ but $u^i \neq 0$ then

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If $A_{u^i} = 0$ but $u^i \neq 0$ then

there is $v, w \in L^2(\Omega) \setminus \{0\}$

$$(\Delta + k^2)v = 0, \qquad \Omega \tag{1}$$
$$(\Delta + k^2(1 + V))u = 0, \qquad \Omega \tag{2}$$
$$u - v \in H_0^2(\Omega). \tag{3}$$

If (1)–(3) has a solution then k^2 is an interior transmission eigenvalue (ITE).

Interior transmission eigenvalues VS sampling methods

Recall: $A_{u^i} = 0, \quad u^i \neq 0 \quad \implies \quad k^2$ ITE

Sampling method users avoid ITE's

Are they too careful?

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$\implies \exists u^i \neq 0, \ A_{u^i} = 0$

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▶ Regge, Newton, Sabatier, Grinevich, Manakov, Novikov
50's – 90's: radial potentials transparent at a fixed k^2 i.e.
$\implies A_{u^i} = 0 \ \forall u^i$

Corner scattering

Theorem (B.–Päivärinta–Sylvester CMP 14)

The potential $V = \chi_{[0,\infty[^n} \varphi$, $\varphi(0) \neq 0$ always scatters.

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The potential $V = \chi_{[0,\infty[^n}\varphi$, $\varphi(0) \neq 0$ always scatters.

For *any* incident wave $u^i \neq 0$ we have $A_{u^i} \neq 0$.

Some follow-up corner scattering results

- ▶ Päivärinta–Salo–Vesalainen: 2D any angle, 3D almost any spherical cone
- ▶ Hu–Salo–Vesalainen: smoothness reduction, new arguments, *polygonal scatterer probing*
- ▶ Elschner–Hu: 3D any domain having two faces meet at an angle, and also curved edges
- ▶ Liu–Xiao: electromagnetic waves

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Injectivity of support probing:

Theorem (HSV+EH)

Let P, P' be convex polyhedra and $V = \chi_P \varphi$, $V = \chi_{P'} \varphi'$ for admissible functions φ, φ'. Then

$$P \neq P' \implies A_{u^i} \neq A'_{u^i} \quad \forall u^i \neq 0$$

Any *single* incident wave determines P in the class of polyhedral penetrable scatterers.

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Any *single* incident wave determines P in the class of polyhedral penetrable scatterers.
However Ikehata's enclosure method gives roughly the same!

My work while in Hong Kong

Stability of polygonal scatterer probing
Non-vanishing total wave

Theorem (B., Liu, preprint)

Let u^i be an incident wave and let $V = \chi_P \varphi$, $V' = \chi_{P'} \varphi'$ be admissible with $|u|, |u'| \neq 0$ in B_R. If

$$\left\| A_{u^i} - A'_{u^i} \right\|_{L^2(\mathbb{S}^{n-1})} < \varepsilon$$

then

$$d_H(P, P') \leq C(\ln \ln \left\| A_{u^i} - A'_{u^i} \right\|_2^{-1})^{-\eta}$$

for some $\eta > 0$.

Lower bound for far-field pattern
Arbitrary Herglotz wave

Theorem (B., Liu, JFA 2017)

Let u^i be a normalized Herglotz wave,

$$u^i(x) = \int_{\mathbb{S}^{n-1}} e^{ik\theta \cdot x} g(\theta) d\sigma(\theta), \qquad \|g\|_{L^2(\mathbb{S}^{n-1})} = 1,$$

and let $V = \chi_P \varphi$ be admissible. Then

$$\|A_{u^i}\|_{L^2(\mathbb{S}^{n-1})} \geq C_{\|P_N\|, V} > 0$$

where the Taylor expansion of u^i at the corner x_c begins with P_N, and $\|P_N\| = \int_{\mathbb{S}^{n-1}} |P_N(\theta)| \, d\sigma(\theta)$.

Mistake?

F. Cakoni: "Incident waves that approximate transmission eigenfunctions produce arbitrarily small far-field patterns."

From apparent contradiction to inspiration

Theorem (B., Liu, + B., Li, Liu, Wang, JFA + IP 2017 + preprint)

Let the potential $V = \chi_\Omega \varphi$ be admissible. Let $v, w \neq 0$ be transmission eigenfunctions:

$$(\Delta + k^2)v = 0, \qquad \Omega$$
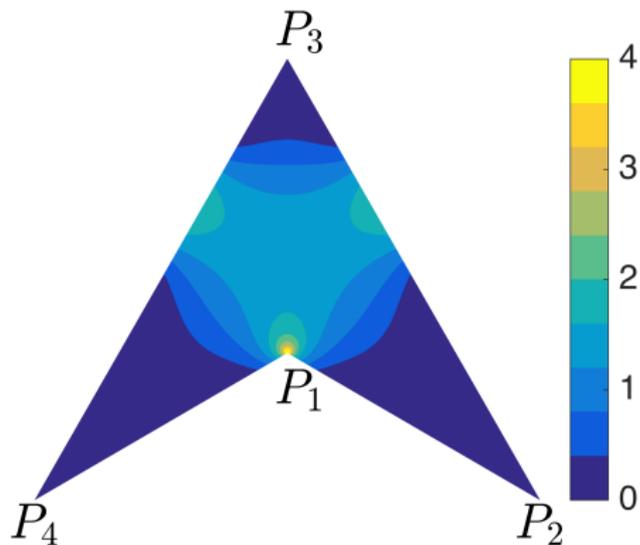$$(\Delta + k^2(1 + V))w = 0, \qquad \Omega$$
$$w - v \in H_0^2(\Omega).$$

Under C^α-smoothness of v near x_c, we have

$$v(x_c) = w(x_c) = 0$$

at every corner point x_c of Ω.

Transmission eigenfunction localization

Piecewise constant recovery

Injectivity of piecewise constant potential probing:

Theorem (B., Liu, preprint)

Let Σ_j, $j = 1, 2, \ldots$ be bounded convex polyhedra in an admissible geometric arrangement (think pixels/voxels) and $V = \sum_j V_j \chi_{\Sigma_j}$, $V = \sum_j V_j' \chi_{\Sigma_j}$ for constants $V_j, V_j' \in \mathbb{C}$. Then

$$V \neq V' \quad \implies \quad A_{u^i} \neq A_{u^i}' \quad \forall u^i(x) = e^{ik\theta \cdot x}$$

if $k > 0$ small or $|u| + |u'| \neq 0$ at each vertex.

A *single* incident plane wave determines V in the class of discretized penetrable scatterers.

Proof sketch

Integration by parts

$$k^2 \int_\Omega (V - V')u'u_0\,dx = \int_{\partial\Omega} \left((u - u')\partial_\nu u_0 - u_0\partial_\nu(u - u')\right)dx$$

if $(\Delta + k^2(1 + V))u_0 = 0$ in Ω.

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Simple case: $\Omega = B(0, \varepsilon) \cap \Sigma_j$ with $\Sigma_j = \,]0, 1[^n$

$$
\begin{aligned}
u'(x) &= u'(0) + u'_r(x) & u' \in H^2 \hookrightarrow C^{1/2} \\
u_0(x) &= e^{\rho \cdot x}(1 + \psi(x)) & \text{CGO} \\
(V - V')(x) &= V_j - V'_j & \text{piecewise constant}
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Hölder estimates give

$$C \left| (V_j - V'_j)u'(0) \right| |\rho|^{-n} \leq \left| (V_j - V'_j)u'(0) \int_{[0,\infty[^n} e^{\rho \cdot x} dx \right| \leq C |\rho|^{-n-\delta}$$
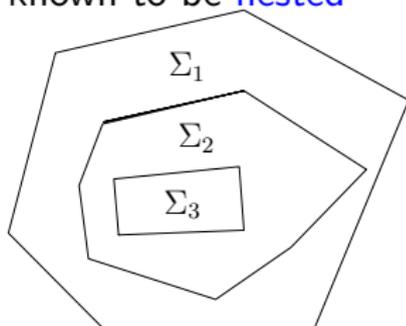
if $\|\psi\|_p \leq C |\rho|^{-n/p-\varepsilon}$.

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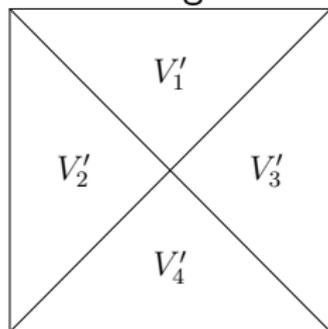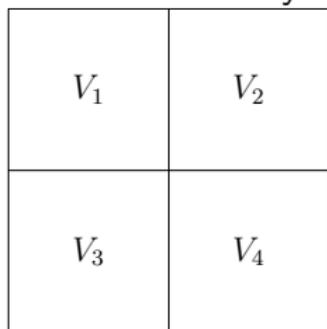
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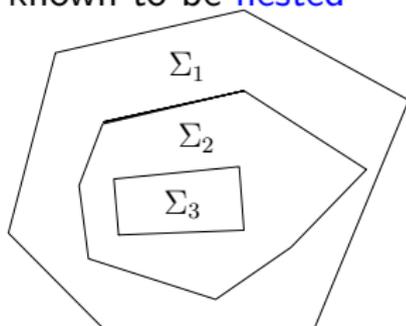
- method cannot yet be shown to distinguish between

Non-scattering
Inverse source problem

$$(\Delta + k^2)u = f, \qquad \lim_{r \to \infty} (\partial_r - ikr)\, u = 0$$

can one have $f \neq 0$ but $u_\infty = 0$?

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Recall:

$$u_\infty(\theta) = c_{k,n}\hat{f}(k\theta).$$

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Yes: let

$$f(x) = \begin{cases} 1, & |x| < r_0 \\ 0, & |x| \geq r_0 \end{cases}$$

where $r_0 > 0$.

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Yes: let
$$f(x) = \begin{cases} 1, & |x| < r_0 \\ 0, & |x| \geq r_0 \end{cases}$$

where $r_0 > 0$. Then

$$u_\infty(\theta) = c_{k,n} \hat{f}(k\theta) = c'_{k,n} J_{n/2}(kr_0) = 0$$

if kr_0 is a zero of the Bessel function of order $n/2$.

Always scattering

A *small* uniform ball always scatters!

Always scattering

A *small* uniform ball always scatters!

Also: any small scatterer always scatters!

Theorem
Let $n \geq 2$, $R_m, k \in \mathbb{R}_+$, $0 \leq \alpha \leq 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain of diameter at most R_m and whose complement is connected. Then there exists $C = C(k, R_m, n) > 0$ such that if $\varphi \in C^\alpha(\overline{\Omega})$ and Ω satisfy

$$\left(\operatorname{diam}(\Omega) \right)^\alpha \leq C \frac{\sup_{\partial\Omega} |\varphi|}{\|\varphi\|_{C^\alpha(\overline{\Omega})}},$$

then the source $f = \chi_\Omega \varphi$ radiates a non-zero far-field pattern at wavenumber k.

Always scattering

Proof.

Suppose $(\Delta + k^2)u = \chi_\Omega \varphi$ and $u_\infty = 0$. Then $u_{|\Omega^\complement} = 0$, so $u_{|\Omega} \in H_0^2(\Omega)$ and $(\Delta + k^2)u = \varphi$ in Ω.

Always scattering

Proof.
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$u_{|\Omega} \in H_0^2(\Omega)$ and $(\Delta + k^2)u = \varphi$ in Ω.
Set $g = \varphi - k^2 u$. Then elliptic regularity implies $g \in C^\alpha(\overline{\Omega})$ with
$\|g\|_{C^\alpha} \leq C(n, k, R_m) \|\varphi\|_{C^\alpha}$. Moreover

$$\int_\Omega g(x)dx = \int_\Omega 1 \cdot \Delta u \, dx = 0$$

because $u = \partial_\nu u = 0$ in $\partial\Omega$.

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Hence

$$|\varphi(p)|\, m(\Omega) \le \|g\|_{C^\alpha} \int_\Omega |x - p|^\alpha \, dx \le \|g\|_{C^\alpha}\, m(\Omega)\big(\operatorname{diam}(\Omega)\big)^\alpha.$$

Always scattering

Is smallness the true cause for non-scattering?

Always scattering
High curvature case

Is smallness the true cause for non-scattering?

No: high curvature!

Theorem
Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain. Assume that $p \in \partial\Omega$ is an admissible K-curvature point. Also assume p connected to infinity outside of Ω. Consider the source $\chi_\Omega \varphi$, $\varphi \in C^\alpha(\mathbb{R}^n)$. If

$$|\varphi(p)| \geq \mathcal{C}(\ln K)^{(n+3)/2} K^{-\delta}$$

then the source scatters a non-zero far-field pattern at wavenumber k.
Here $\delta > 0$ depends on the geometric parameters, and $\mathcal{C}$ depends on the a-priori parameters of the K-curvature point, the wavenumber k, the upper bound for the diameter of Ω, and the upper bound for $\|\varphi\|_{C^\alpha}$.

Admissible K-curvature point
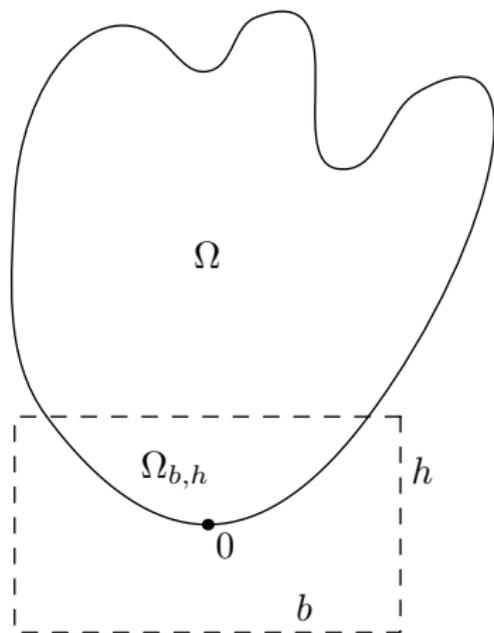
Inverse source problem, Schiffer's problem

$$(\Delta + k^2)u = f = \chi_\Omega \varphi, \qquad \lim_{r \to \infty} (\partial_r - ikr)u = 0$$

Can $u_\infty(\theta) = c\hat{f}(k\theta)$ determine Ω given a fixed k?

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Can $u_\infty(\theta) = c\hat{f}(k\theta)$ determine Ω given a fixed k?

Unique determination:

- $u_\infty = u'_\infty \Longrightarrow \Omega = \Omega'$ for convex polyhedral shapes (corner scattering),

- $u_\infty = u'_\infty \Longrightarrow \Omega \approx \Omega'$ for convex polyhedral shapes whose corners have been smoothened to admissible K-curvature points (high curvature scattering),

- $u_\infty = u'_\infty \Longrightarrow \Omega \approx \Omega'$ for well-separated collections of small scatterers (small source scattering).

Thank you for your attention!