Topics in Corner Scattering:
Non-Scattering Waves, Potential Probing with a Single Incident Wave, and the Interior Transmission Problem

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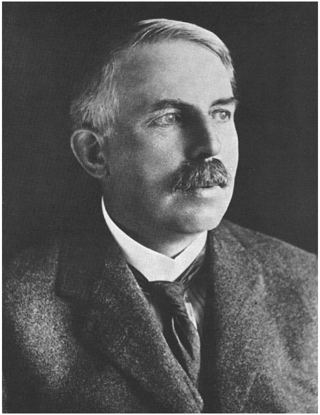
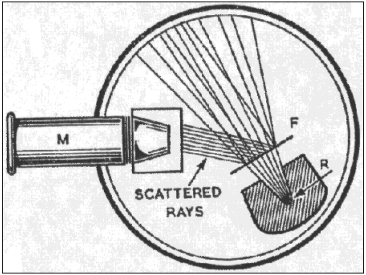
Scattering theory

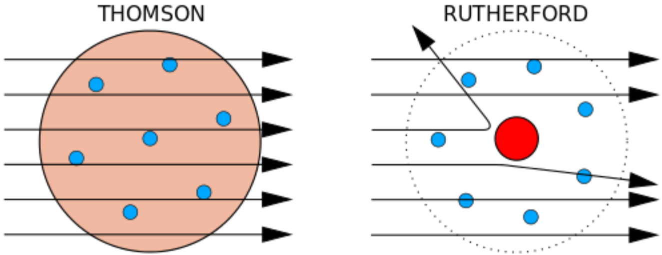
Lord Rutherford's gold-foil experiment





Scattering theory

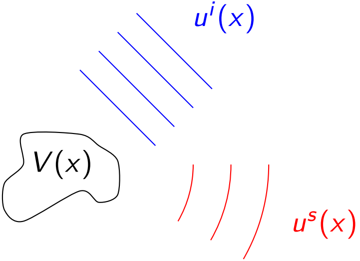
Rutherford experiment's conclusions



measurement + a-priori information = conclusion

Scattering theory

Fixed frequency scattering



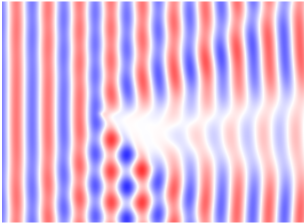
The total wave u satisfies

$$\big(\Delta + k^2(1+V)\big)u = 0,$$

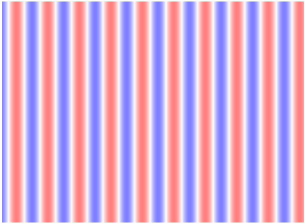
V models a perturbation of the background,

$$u = \underbrace{u^i(x)}_{\text{incident wave}} + \underbrace{u^s(x)}_{\text{scattered wave}}$$

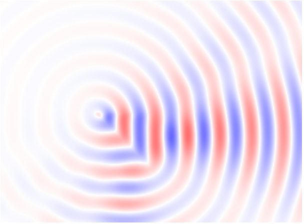
Scattering theory



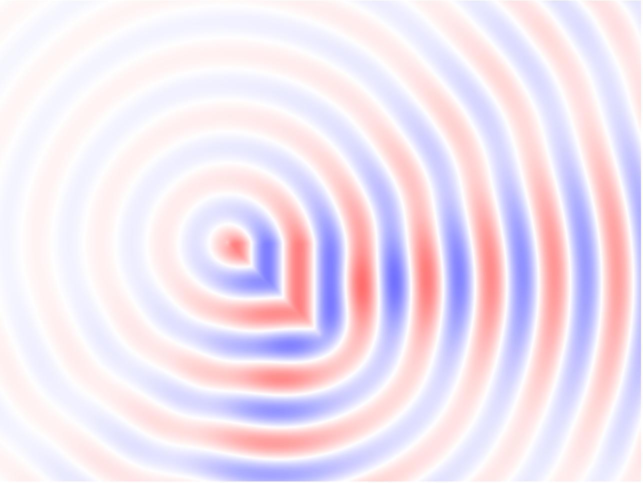
$$=$$



$$+$$



Mathematical scattering theory: measurements



Measurement: A_{u^i} is the far-field pattern of the scattered wave

$$u^s(x) = \frac{e^{ik|x|}}{|x|^{(n-1)/2}} A_{u^i}\left(\frac{x}{|x|}\right) + \mathcal{O}\left(\frac{1}{|x|^{n/2}}\right)$$

Inverse problem

Given the map $u^i \mapsto A_{u^i}$, recover V or its support Ω.

Early methods ($<$ 85')

- optimization and minimization methods

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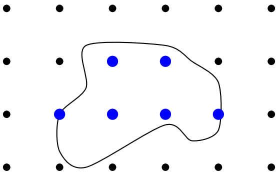
Early methods (< 85')

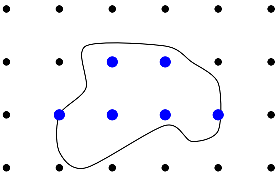
- optimization and minimization methods

Sampling methods

- gives condition on measurements for $x \in \operatorname{supp} V$
- compared to before: *fast! works reliably!*

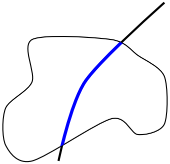


Sampling methods



- 96 Colton – Kirsh: linear sampling method (points)

Sampling methods



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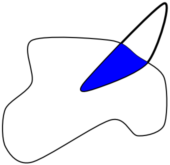

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- . . . Luke, Potthast, Sylvester, Kusiak: range test, no response test (sets)

Factorization method

Sampling methods gave only[1] *sufficient* conditions for $x \in \operatorname{supp} V$.

[1] except Ikehata's probing method

Factorization method

Kirsch 90's, Grinberg 00's: factorization method. Gives *necessary and sufficient* conditions.

Factorization method

Idea:

$$u^i(x) = \int_{\mathbb{S}^{n-1}} e^{ik\theta\cdot x} g(\theta) d\sigma(\theta), \qquad g \in L^2(\mathbb{S}^{n-1})$$

$$u^s(x) = \frac{e^{ik|x|}}{|x|^{(n-1)/2}} A_g\left(\frac{x}{|x|}\right) + \mathcal{O}\left(\frac{1}{|x|^{n/2}}\right)$$

the far-field operator

$$F : L^2(\mathbb{S}^{n-1}) \to L^2(\mathbb{S}^{n-1}), \qquad Fg = A_g$$

is factored

$$F = G\, T\, G^*$$

G compact, T isomorphism. The range of G can be characterized and gives $\operatorname{supp} V$.

Everything solved?

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NO!

If $\ker F \neq \{0\}$ then the above methods fail!

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$\exists g \in \ker F$ implies $\exists v : \Omega \to \mathbb{C}$

$$\begin{aligned} (\Delta + k^2)v = 0, &\qquad \Omega \\ (\Delta + k^2(1+V))u = 0, &\qquad \Omega \\ u - v \in H_0^2(\Omega) & \end{aligned}$$

k^2 is an interior transmission eigenvalue (ITE)

Kernel of scattering operator

Let w^i be the incident wave

$$w^i(x) = \int_{\mathbb{S}^{n-1}} e^{ik\theta\cdot x} g(\theta) d\sigma(\theta)$$

and assume $g \in \ker F$. Then $A_g \equiv 0$ for the scattered wave w^s.

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Rellich's lemma and unique continuation imply $w^s(x) = 0$ for $x \in \mathbb{R}^n \setminus \mathrm{supp}\, V$.

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Hence $v = w^i$ and $u = w^i + w^s$ solve the interior transmission problem.

Some ITE history

- 86', 88' **Kirsch, Colton–Monk**: ITE problem posed
- 89', 91' **Colton–Kirsch–Päivärinta, Rynne–Sleeman**: discreteness of ITE

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- 07', 09' **Cakoni–Colton–Monk, Cakoni–Colton–Haddar**: qualitative information about V from ITE's
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- 10'+: EXPLOSION OF INTEREST
- interest shifting to "Steklov eigenvalues"

Interior transmission eigenvalues VS sampling methods

Recall: $\ker F \neq \{0\} \quad \Longrightarrow \quad k^2$ ITE

Sampling method users avoid ITE's

Are they too careful?

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- Colton–Monk 88: $\operatorname{supp} V$ compact, V radial, k^2 ITE $\Longrightarrow \ker F \neq \{0\}$
- Regge, Newton, Sabatier, Grinevich, Manakov, Novikov 50's – 90's: radial potentials transparent at a fixed k^2 i.e. $\Longrightarrow \ker F = L^2(\mathbb{S}^{n-1})$

Corner scattering

- B.–Päivärinta–Sylvester 14: $V = \chi_{[0,\infty[^n}\varphi$, $\varphi(0) \neq 0$ always scatters, *despite having interior transmission eigenvalues*

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$$k^2 \text{ ITE } \textit{and } \ker F = \{0\}$$

Proof sketch

Rellich's theorem and unique continuation imply

$$k^2 \int V u^i u_0 dx = 0$$

if $(\Delta + k^2(1+V))u_0 = 0$ near $\operatorname{supp} V$.

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In simple case

$$\begin{aligned} u^i(x) &= u^i(0) + u^i_r(x) \\ u_0(x) &= e^{\rho\cdot x}(1+\psi(x)) \\ V(x) &= \chi_{[0,\infty[^n}(x)(\varphi(0) + \varphi_r(x)) \end{aligned}$$

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Hölder estimates give

$$C\left|\varphi(0)u^i(0)\right| |\rho|^{-n} \leq \left|\varphi(0)u^i(0)\int_{[0,\infty[^n} e^{\rho\cdot x} dx\right| \leq C|\rho|^{-n-\delta}$$

if $\|\psi\|_p \leq C|\rho|^{-n/p-\varepsilon}$.

Newer corner scattering results

- Päivärinta–Salo–Vesalainen: 2D any angle, 3D almost any spherical cone
- Hu–Salo–Vesalainen: smoothness reduction, new arguments, *polygonal scatterer probing*
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Injectivity of support probing:

Theorem
Let P, P' be convex polygons and $V = \chi_P \varphi$, $V = \chi_{P'} \varphi'$ for admissible functions φ, φ'. Then

$$P \neq P' \quad \Longrightarrow \quad F_V(g) \neq F_{V'}(g) \quad \forall g \neq 0$$

Any *single* incident wave determines P in the class of polygonal penetrable scatterers.

Stability of polygonal scatterer probing

Theorem (B., Liu, preprint)

Let u^i be an incident wave with $u^i(x) \neq 0$ and let $V = \chi_P \varphi$, $V' = \chi_{P'} \varphi'$ be admissible. If

$$\left\| A_{u^i} - A'_{u^i} \right\|_{L^2(\mathbb{S}^{n-1})} < \varepsilon$$

then

$$d_H(P, P') \leq C \left(\ln \ln \left\| A_{u^i} - A'_{u^i} \right\|_2^{-1} \right)^{-\eta}$$

for some $\eta > 0$.

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Probing impenetrable scatterers with few waves: J. Li, H. Liu, M. Petrini, L. Rondi, J. Xiao, Y. Wang . . .

Proof structure

- Quantify everything in corner scattering proofs
- Use fact that total wave does not vanish in domain of interest
- Propagate smallness from ∞ to $P \cup P'$

Far-field to near-field to boundary

- A quantitative version of Rellich's theorem + unique continuation

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- A quantitative version of Rellich's theorem + unique continuation
- Isakov-type theorem + chain of balls + smoothness of u^s

Proposition

Let u^s and u'^s be the scattered waves caused by u^i. If $Q = \mathrm{ch}(P \cup P')$ and $\|u^s_\infty - u'^s_\infty\|_{L^2(\mathbb{S}^{n-1})} < \varepsilon$ then

$$\sup_{\partial Q} |u - u'| + |\nabla(u - u')| \leq C \left(\ln \ln \|u^s_\infty - u'^s_\infty\|_2^{-1}\right)^{-1/2}.$$

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- $(\Delta + k^2)(u^s - u'^s) = f$ with f supported on $P \cup P'$.
- second logarithm arises from continuing $u^s - u'^s$ from the set where $f \equiv 0$ to its boundary by smoothness.

Boundary to neighbourhood of corner

Let $Q = \mathrm{ch}(P \cap P')$, $Q_h = Q \cap B(x_c, h)$, $h < d(P, P')$. If u_0 is a CGO solution for V then

$$k^2 \int_{Q_h} V u' u_0 dx = \int_{\partial Q_h} (u_0 \partial_\nu (u' - u) - (u' - u) \partial_\nu u_0) d\sigma.$$

Estimates

$$k^2 \int_{Q_h} V u' u_0 dx = \int_{\partial Q_h} (u_0 \partial_\nu (u' - u) - (u' - u) \partial_\nu u_0) d\sigma.$$

Split LHS as before and use $u' \neq 0$ everywhere in Q_h. CGO and Hölder estimates give

$$C \leq |\rho|^n \left| \int_{\mathfrak{P}} e^{\rho \cdot x} dx \right| \leq h^{-1} |\rho|^{-\delta} + |\rho|^3 \left(\ln \ln \|u^s_\infty - u'^s_\infty\|_2^{-1}\right)^{-1/2}.$$

The claim

$$d_H(P, P') \leq C (\ln \ln \|A_{u^i} - A'_{u^i}\|_2^{-1})^{-\eta}$$

follows since $h < d(P, P')$.

More recent work: lower bound for far-field pattern

Theorem (B., Liu, preprint)

Let u^i be a normalized Herglotz wave,

$$u^i(x) = \int_{\mathbb{S}^{n-1}} e^{ik\theta\cdot x} g(\theta) d\sigma(\theta), \qquad \|g\|_{L^2(\mathbb{S}^{n-1})} = 1,$$

and let $V = \chi_P \varphi$ be admissible.

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and let $V = \chi_P \varphi$ be admissible. Then

$$\|u^s_\infty\|_{L^2(\mathbb{S}^{n-1})} \geq C_{\|P_N\|,V} > 0$$

where the Taylor expansion of u^i at the corner x_c begins with P_N, and $\|P_N\| = \int_{\mathbb{S}^{n-1}} |P_N(\theta)|\, d\sigma(\theta)$.

Mistake?

F. Cakoni: “Incident waves that approximate transmission eigenfunctions produce arbitrarily small far-field patterns.”

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So, mistake in our proof?
– No: $C = C_{\|P_N\|}$, so the bound becomes arbitrarily small for incident waves that have small value at the corner.

From contradiction to inspiration

Theorem (B., Liu, preprint)

Let the potential $V = \chi_P \varphi$ *be admissible and* $P \subset \Omega$. *Let* v *be a transmission eigenfunction*

$$
\begin{aligned}
(\Delta + k^2)v = 0, &\qquad \Omega \\
(\Delta + k^2(1+V))w = 0, &\qquad \Omega \\
w - v \in H_0^2(\Omega), &\qquad \|v\|_{L^2(\Omega)} = 1.
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$$\begin{aligned} (\Delta + k^2)v &= 0, && \Omega \\ (\Delta + k^2(1+V))w &= 0, && \Omega \\ w - v &\in H_0^2(\Omega), && \|v\|_{L^2(\Omega)} = 1. \end{aligned}$$

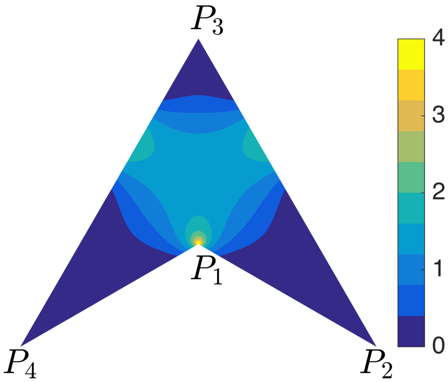
If v can be approximated by a sequence of Herglotz waves with *uniformly L^2-bounded kernels g*, then

$$\lim_{r \to 0} \frac{1}{m(B(x_c, r))} \int_{B(x_c,r)} |v(x)|\, dx = 0$$

at every corner point x_c of $\operatorname{supp} V$.

Transmission eigenfunction localization

Ongoing numerical investigation with Y. Wang and H. Liu:



Conclusions

- sampling methods fail when no scattering
- avoid transmission eigenvalues $\Longrightarrow$ have scattering
- corners always scatter — despite having transmission eigenvalues
- single wave inverse scattering: polygonal support uniqueness
- lower bound for far-field pattern
- transmission eigenfunction localization

Thank you for your attention!