Applications of corner scattering: intrinsic properties of transmission eigenfunctions and single wave probing

Eemeli Blåsten

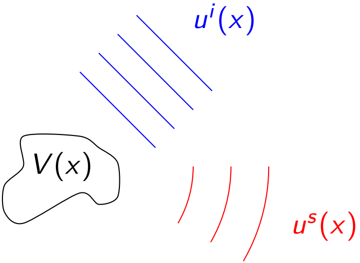
Institute for Advanced Study,
The Hong Kong University of Science and Technology

School of Mathematical Sciences, Fudan University

Shanghai, December 5, 2017

Scattering theory

Fixed frequency scattering



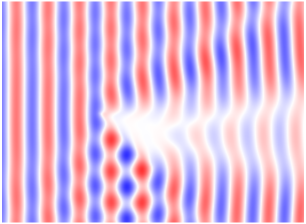
The total wave u satisfies

$$\big(\Delta + k^2(1+V)\big)u = 0,$$

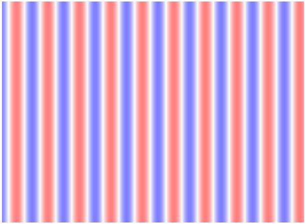
V models a perturbation of the background,

$$u = \underbrace{u^i(x)}_{\text{incident wave}} + \underbrace{u^s(x)}_{\text{scattered wave}}$$

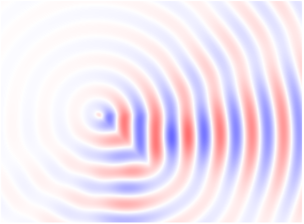
Scattering theory



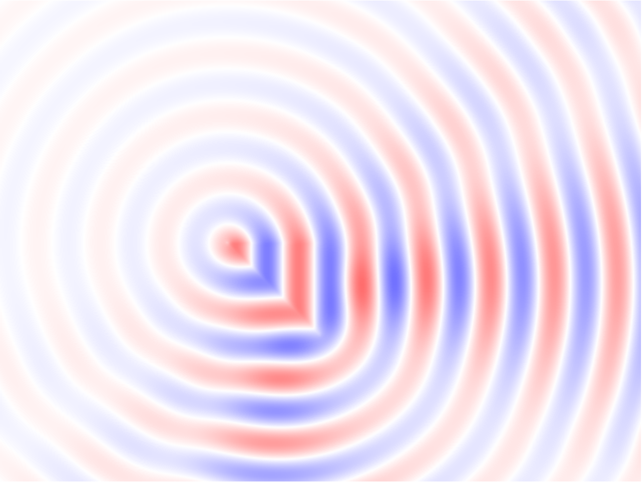
$=$



$+$



Mathematical scattering theory: measurements



Measurement: A_{u^i} is the far-field pattern of the scattered wave

$$u^s(x) = \frac{e^{ik|x|}}{|x|^{(n-1)/2}} A_{u^i}\left(\frac{x}{|x|}\right) + \mathcal{O}\left(\frac{1}{|x|^{n/2}}\right)$$

Inverse problem

Given the map $u^i \mapsto A_{u^i}$, recover V or its support Ω.

Early methods ($<$ 85')

- optimization and minimization methods

Inverse problem

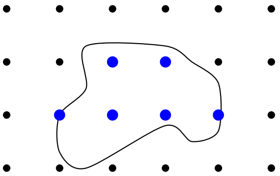
Given the map $u^i \mapsto A_{u^i}$, recover V or its support Ω.

Early methods ($<$ 85')

- optimization and minimization methods

Sampling methods

- looks for $\operatorname{supp} V$
- compared to before: *fast! works reliably!*



Inverse problem

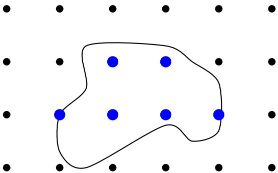
Given the map $u^i \mapsto A_{u^i}$, recover V or its support Ω.

Early methods ($<$ 85')

- optimization and minimization methods

Sampling methods

- looks for $\operatorname{supp} V$
- compared to before: *fast! works reliably!*

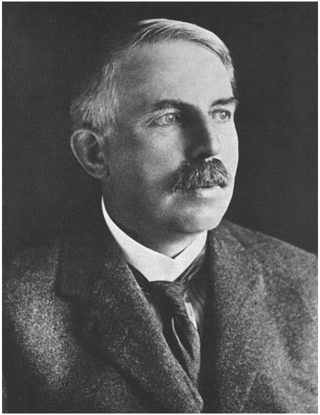


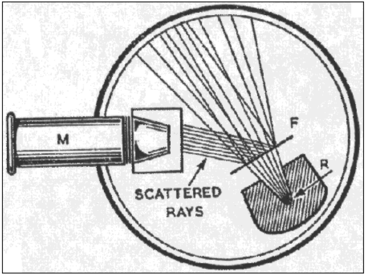
Methods based on Sylvester–Uhlmann 87 CGO solutions

- countable family $(u^i_j, A_{u^i_j})_{j=1}^{\infty}$ determines V

What about in physics?

Lord Rutherford's gold-foil experiment

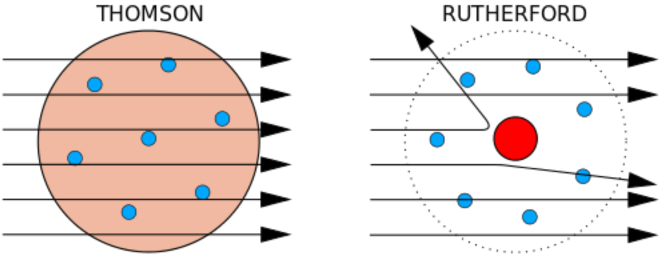




Single incident wave

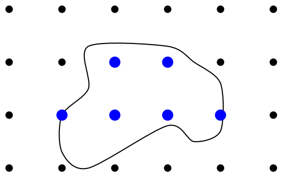
Scattering theory

Rutherford experiment's conclusions



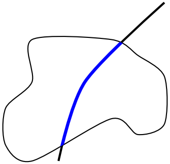
measurement + *a-priori information* = conclusion

Sampling methods



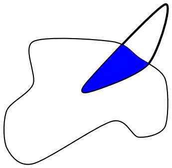
- 96 Colton – Kirsh: linear sampling method (points)

Sampling methods



- 96 Colton – Kirsh: linear sampling method (points)
- 98 Ikehata: probing method (curve)

Sampling methods



- 96 Colton – Kirsh: linear sampling method (points)
- 98 Ikehata: probing method (curve)
- . . . Luke, Potthast, Sylvester, Kusiak: range test, no response test (sets)

Factorization method

Sampling methods gave only[1] *sufficient* conditions for $x \in \operatorname{supp} V$.

[1] except Ikehata's probing method

Factorization method

Kirsch 90's, Grinberg 00's: factorization method. Gives *necessary and sufficient* conditions.

Factorization method

Idea:

$$u^i(x) = \int_{\mathbb{S}^{n-1}} e^{ik\theta\cdot x} g(\theta) d\sigma(\theta), \qquad g \in L^2(\mathbb{S}^{n-1})$$

$$u^s(x) = \frac{e^{ik|x|}}{|x|^{(n-1)/2}} A_g\left(\frac{x}{|x|}\right) + \mathcal{O}\left(\frac{1}{|x|^{n/2}}\right)$$

the far-field operator

$$F : L^2(\mathbb{S}^{n-1}) \to L^2(\mathbb{S}^{n-1}), \qquad Fg = A_g$$

is factored

$$F = G\, T\, G^*$$

G compact, T isomorphism. The range of G can be characterized and gives $\operatorname{supp} V$.

Everything solved?

Everything solved?

NO!

If $\ker F \neq \{0\}$ then the above methods fail!

Everything solved?

NO!

If $\ker F \neq \{0\}$ then the above methods fail!

$\exists g \in \ker F$ implies $\exists v : \Omega \to \mathbb{C}$

$$\begin{aligned} (\Delta + k^2)v &= 0, && \Omega \\ (\Delta + k^2(1+V))u &= 0, && \Omega \\ u - v &\in H_0^2(\Omega), && \|v\|_{L^2(\Omega)} = 1. \end{aligned}$$

k^2 is an interior transmission eigenvalue (ITE)

Kernel of scattering operator

Let w^i be the incident wave

$$w^i(x) = \int_{\mathbb{S}^{n-1}} e^{ik\theta\cdot x} g(\theta) d\sigma(\theta)$$

and assume $g \in \ker F$. Then $A_g \equiv 0$ for the scattered wave w^s.

Kernel of scattering operator

Let w^i be the incident wave

$$w^i(x) = \int_{\mathbb{S}^{n-1}} e^{ik\theta\cdot x} g(\theta) d\sigma(\theta)$$

and assume $g \in \ker F$. Then $A_g \equiv 0$ for the scattered wave w^s.

Rellich's lemma and unique continuation imply $w^s(x) = 0$ for $x \in \mathbb{R}^n \setminus \operatorname{supp} V$.

Kernel of scattering operator

Let w^i be the incident wave

$$w^i(x) = \int_{\mathbb{S}^{n-1}} e^{ik\theta\cdot x} g(\theta) d\sigma(\theta)$$

and assume $g \in \ker F$. Then $A_g \equiv 0$ for the scattered wave w^s.

Rellich's lemma and unique continuation imply $w^s(x) = 0$ for $x \in \mathbb{R}^n \setminus \operatorname{supp} V$.

Hence $v = w^i$ and $u = w^i + w^s$ solve the interior transmission problem.

Some ITE history

- 86', 88' **Kirsch, Colton–Monk**: ITE problem posed
- 89', 91' **Colton–Kirsch–Päivärinta, Rynne–Sleeman**: discreteness of ITE

Some ITE history

- 86', 88' **Kirsch, Colton–Monk**: ITE problem posed
- 89', 91' **Colton–Kirsch–Päivärinta, Rynne–Sleeman**: discreteness of ITE
- 91'–08' NOTHING. . .

Some ITE history

- 86', 88' **Kirsch, Colton–Monk**: ITE problem posed
- 89', 91' **Colton–Kirsch–Päivärinta, Rynne–Sleeman**: discreteness of ITE
- 91'–08' NOTHING...
- 07', 09' **Cakoni–Colton–Monk, Cakoni–Colton–Haddar**: qualitative information about V from ITE's
- 08' **Päivärinta–Sylvester**: existence for general scatterers
- 10' **Cakoni–Gintides–Haddar**: infinitely many ITE's
- 10' **Cakoni–Colton–Haddar**: ITE's can be deduced from far-field data
- 10'+: EXPLOSION OF INTEREST

Some ITE history

- 86', 88' **Kirsch, Colton–Monk**: ITE problem posed
- 89', 91' **Colton–Kirsch–Päivärinta, Rynne–Sleeman**: discreteness of ITE
- 91'–08' NOTHING...
- 07', 09' **Cakoni–Colton–Monk, Cakoni–Colton–Haddar**: qualitative information about V from ITE's
- 08' **Päivärinta–Sylvester**: existence for general scatterers
- 10' **Cakoni–Gintides–Haddar**: infinitely many ITE's
- 10' **Cakoni–Colton–Haddar**: ITE's can be deduced from far-field data
- 10'+: EXPLOSION OF INTEREST
- interest shifting to "Steklov eigenvalues"

Interior transmission eigenvalues VS sampling methods

Recall: $\ker F \neq \{0\} \quad \Longrightarrow \quad k^2$ ITE

Sampling method users avoid ITE's

Are they too careful?

Interior transmission eigenvalues VS sampling methods

Recall: $\ker F \neq \{0\} \quad \Longrightarrow \quad k^2$ ITE

Sampling method users avoid ITE's

Are they too careful?

- Colton–Monk 88: $\operatorname{supp} V$ compact, V radial, k^2 ITE $\Longrightarrow \ker F \neq \{0\}$

Interior transmission eigenvalues VS sampling methods

Recall: $\ker F \neq \{0\} \quad \Longrightarrow \quad k^2$ ITE

Sampling method users avoid ITE's

Are they too careful?

- Colton–Monk 88: $\operatorname{supp} V$ compact, V radial, k^2 ITE $\Longrightarrow \ker F \neq \{0\}$
- Regge, Newton, Sabatier, Grinevich, Manakov, Novikov 50's – 90's: radial potentials transparent at a fixed k^2 i.e. $\Longrightarrow \ker F = L^2(\mathbb{S}^{n-1})$

Corner scattering

Theorem (B.–Päivärinta–Sylvester 14)

The potential $V = \chi_{[0,\infty[^n}\varphi$, $\varphi(0) \neq 0$ *always scatters.*

Corner scattering

Theorem (B.–Päivärinta–Sylvester 14)
The potential $V = \chi_{[0,\infty[^n}\varphi$, $\varphi(0) \neq 0$ *always scatters.*

For *any* incident wave $u^i \neq 0$ we have $A_{V,u^i} \neq A_{0,u^i}$.

Proof sketch

Rellich's theorem and unique continuation imply

$$k^2 \int V u^i u_0 dx = 0$$

if $(\Delta + k^2(1+V))u_0 = 0$ near $\operatorname{supp} V$.

Proof sketch

Rellich's theorem and unique continuation imply

$$k^2 \int V u^i u_0 dx = 0$$

if $(\Delta + k^2(1+V))u_0 = 0$ near $\operatorname{supp} V$.
In simple case

$$\begin{aligned} u^i(x) &= u^i(0) + u^i_r(x) \\ u_0(x) &= e^{\rho \cdot x}(1 + \psi(x)) \\ V(x) &= \chi_{[0,\infty[^n}(x)(\varphi(0) + \varphi_r(x)) \end{aligned}$$

Proof sketch

Rellich's theorem and unique continuation imply

$$k^2 \int V u^i u_0 dx = 0$$

if $(\Delta + k^2(1+V))u_0 = 0$ near $\operatorname{supp} V$.
In simple case

$$\begin{aligned} u^i(x) &= u^i(0) + u^i_r(x) \\ u_0(x) &= e^{\rho\cdot x}(1+\psi(x)) \\ V(x) &= \chi_{[0,\infty[^n}(x)(\varphi(0)+\varphi_r(x)) \end{aligned}$$

Hölder estimates give

$$C\left|\varphi(0)u^i(0)\right| |\rho|^{-n} \leq \left|\varphi(0)u^i(0)\int_{[0,\infty[^n} e^{\rho\cdot x}dx\right| \leq C|\rho|^{-n-\delta}$$

if $\|\psi\|_p \leq C|\rho|^{-n/p-\varepsilon}$.

Some newer corner scattering results

- Päivärinta–Salo–Vesalainen: 2D any angle, 3D almost any spherical cone
- Hu–Salo–Vesalainen: smoothness reduction, new arguments, *polygonal scatterer probing*
- Elschner–Hu: 3D any domain having two faces meet at an angle
- Liu–Xiao: electromagnetic waves

Some newer corner scattering results

- Päivärinta–Salo–Vesalainen: 2D any angle, 3D almost any spherical cone
- Hu–Salo–Vesalainen: smoothness reduction, new arguments, *polygonal scatterer probing*
- Elschner–Hu: 3D any domain having two faces meet at an angle
- Liu–Xiao: electromagnetic waves

Injectivity of support probing:

Theorem (HSV+EH)

Let P, P' be convex polyhedra and $V = \chi_P \varphi$, $V = \chi_{P'} \varphi'$ for admissible functions φ, φ'. Then

$$P \neq P' \quad \Longrightarrow \quad A_{u^i} \neq A'_{u^i} \quad \forall u^i \neq 0$$

Any *single* incident wave determines P in the class of polyhedral penetrable scatterers.

Stability of polygonal scatterer probing

Non-vanishing total wave

Theorem (B., Liu, preprint)

Let u^i *be an incident wave and let* $V = \chi_P \varphi$, $V' = \chi_{P'} \varphi'$ *be admissible with* $|u|, |u'| \neq 0$ *in* B_R. *If*

$$\left\| A_{u^i} - A'_{u^i} \right\|_{L^2(\mathbb{S}^{n-1})} < \varepsilon$$

then

$$d_H(P, P') \leq C\left(\ln \ln \left\| A_{u^i} - A'_{u^i} \right\|_2^{-1}\right)^{-\eta}$$

for some $\eta > 0$.

Stability of polygonal scatterer probing

Non-vanishing total wave

Theorem (B., Liu, preprint)

Let u^i be an incident wave and let $V = \chi_P \varphi$, $V' = \chi_{P'} \varphi'$ be admissible with $|u|, |u'| \neq 0$ in B_R. If

$$\left\| A_{u^i} - A'_{u^i} \right\|_{L^2(\mathbb{S}^{n-1})} < \varepsilon$$

then

$$d_H(P, P') \leq C \big(\ln \ln \left\| A_{u^i} - A'_{u^i} \right\|_2^{-1} \big)^{-\eta}$$

for some $\eta > 0$.

Related work: Probing impenetrable scatterers with few waves:
J. Li, H. Liu, M. Petrini, L. Rondi, J. Xiao, Y. Wang ...

Lower bound for far-field pattern

Arbitrary Herglotz wave

Theorem (B., Liu, JFA 2017)

Let u^i be a normalized Herglotz wave,

$$u^i(x) = \int_{\mathbb{S}^{n-1}} e^{ik\theta\cdot x} g(\theta) d\sigma(\theta), \qquad \|g\|_{L^2(\mathbb{S}^{n-1})} = 1,$$

and let $V = \chi_P \varphi$ be admissible. Then

$$\|A_{u^i}\|_{L^2(\mathbb{S}^{n-1})} \geq C_{\|P_N\|, V} > 0$$

where the Taylor expansion of u^i at the corner x_c begins with P_N, and $\|P_N\| = \int_{\mathbb{S}^{n-1}} |P_N(\theta)| \, d\sigma(\theta)$.

Mistake?

F. Cakoni: "Incident waves that approximate transmission eigenfunctions produce arbitrarily small far-field patterns."

From apparent contradiction to inspiration

Theorem (B., Liu, + B., Li, Liu, Wang, IP + JFA 2017 + preprint)

Let the potential $V = \chi_P \varphi$ be admissible and $P \subset \Omega$. Let v be a transmission eigenfunction:

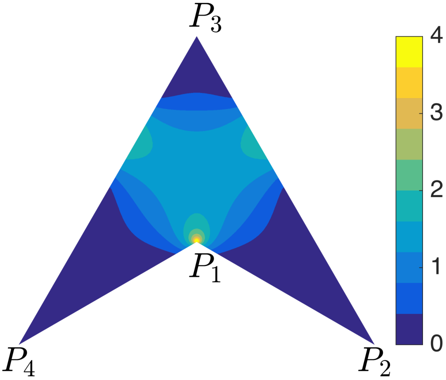
$$\begin{aligned} (\Delta + k^2)v &= 0, & \Omega \\ (\Delta + k^2(1+V))w &= 0, & \Omega \\ w - v \in H_0^2(\Omega)&, & \|v\|_{L^2(\Omega)} = 1. \end{aligned}$$

Under H^2-smoothness of v near x_c, we have

$$\lim_{r\to 0} \frac{1}{m(B(x_c,r))} \int_{B(x_c,r)} |v(x)|\, dx = 0$$

at every corner point x_c of $\operatorname{supp} V$.

Transmission eigenfunction localization



Piecewise constant recovery

Injectivity of piecewise constant potential probing:

Theorem (B., Liu, preprint)

Let Σ_j, $j = 1, 2, \ldots$ *be bounded convex polyhedra in an admissible geometric arrangement (think pixels/voxels) and* $V = \sum_j V_j \chi_{\Sigma_j}$, $V = \sum_j V_j' \chi_{\Sigma_j}$ *for constants* $V_j, V_j' \in \mathbb{C}$. *Then*

$$V \neq V' \quad \Longrightarrow \quad A_{u^i} \neq A'_{u^i} \quad \forall u^i(x) = e^{ik\theta\cdot x}$$

if $k > 0$ *small or* $|u| + |u'| \neq 0$ *at each vertex.*

A *single* incident plane wave determines V in the class of discretized penetrable scatterers.

Proof sketch

Integration by parts

$$k^2 \int_\Omega (V - V')u' u_0 dx = \int_{\partial\Omega} \big((u - u')\partial_\nu u_0 - u_0 \partial_\nu (u - u')\big) dx$$

if $(\Delta + k^2(1 + V))u_0 = 0$ in Ω.

Proof sketch

Integration by parts

$$k^2 \int_\Omega (V - V')u' u_0 dx = \int_{\partial\Omega} \left((u - u')\partial_\nu u_0 - u_0 \partial_\nu (u - u')\right) dx$$

if $(\Delta + k^2(1 + V))u_0 = 0$ in Ω.
Simple case: $\Omega = B(0, \varepsilon) \cap \Sigma_j$ with $\Sigma_j =]0, 1[^n$

$$\begin{aligned} u'(x) &= u'(0) + u'_r(x) && u' \in H^2 \hookrightarrow C^{1/2} \\ u_0(x) &= e^{\rho \cdot x}(1 + \psi(x)) && \text{CGO} \\ (V - V')(x) &= V_j - V'_j && \text{piecewise constant} \end{aligned}$$

Proof sketch

Integration by parts

$$k^2 \int_\Omega (V - V')u' u_0 dx = \int_{\partial\Omega} ((u - u')\partial_\nu u_0 - u_0 \partial_\nu (u - u'))dx$$

if $(\Delta + k^2(1 + V))u_0 = 0$ in Ω.
Simple case: $\Omega = B(0, \varepsilon) \cap \Sigma_j$ with $\Sigma_j =]0, 1[^n$

$$\begin{aligned} u'(x) &= u'(0) + u'_r(x) && u' \in H^2 \hookrightarrow C^{1/2} \\ u_0(x) &= e^{\rho\cdot x}(1 + \psi(x)) && \text{CGO} \\ (V - V')(x) &= V_j - V'_j && \text{piecewise constant} \end{aligned}$$

Hölder estimates give

$$C \left|(V_j - V'_j)u'(0)\right| |\rho|^{-n} \leq \left|(V_j - V'_j)u'(0) \int_{[0,\infty[^n} e^{\rho\cdot x} dx\right| \leq C |\rho|^{-n-\delta}$$

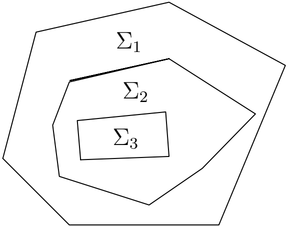
if $\|\psi\|_p \leq C |\rho|^{-n/p-\varepsilon}$.

Generalizations and limitations

- unique determination of corner location *and* value

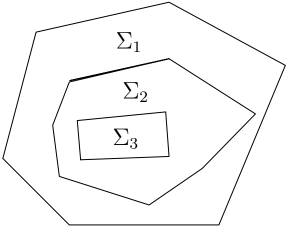
Generalizations and limitations

- unique determination of corner location *and* value
- if Σ_j not known in advance: both $(\Sigma_j)_{j=1}^{\infty}$ and $V = \sum_j V_j \chi_{\Sigma_j}$ uniquely determined by a single measurement if geometry known to be nested

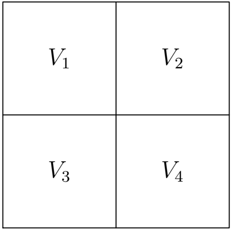


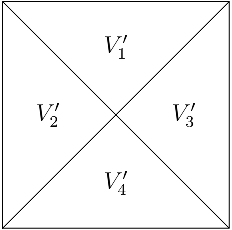
Generalizations and limitations

- unique determination of corner location *and* value
- if Σ_j not known in advance: both $(\Sigma_j)_{j=1}^{\infty}$ and $V = \sum_j V_j \chi_{\Sigma_j}$ uniquely determined by a single measurement if geometry known to be nested



- method cannot yet be shown to distinguish between

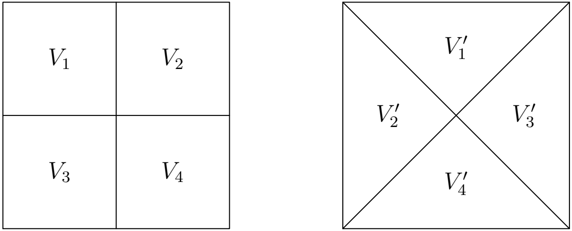




Conclusions

- traditional potential recovery: many probing waves
- physics experiment: one probing wave + much a-priori data
- corners always scatter — even with a "bad" probing wave
- transmission eigenfunctions vanish at corners
- single probing wave
  - polyhedral support uniqueness
  - piecewise constant potential uniqueness
  - some natural questions still unanswered
- if problem is hard, get more a-priori information

Open questions

- split corner 

- show blow-up at non-convex corner (numerical evidence exists)
- study other eigenfunction properties from spectral theory:
  - nodal sets
  - nodal domains (Courant's nodal line theorem: m-th eigenfunction splits domain into $\leq m$ nodal domains).
  - other boundary shapes (ongoing: value bound as a function of curvature)
- actual bounds: if $\left|u^i\right| = \epsilon > 0$ at corners, then what lower bound on $\|A_{u^i}\|$?

Thank you for your attention!