Inverse backscattering with point-source waves

1 History and ordinary backscattering

Scattering theory: Incident wave u^i of single frequency $k \in \mathbb{R}$ given (e.g. plane-wave), the scattered wave is u^s and their sum u is the physical total field. The sign of "$-i\,k$" tells that this is a causal wave ("+" is anticausal)

$$
\begin{aligned}
(-\Delta - k^2)u^i &= 0 \quad && x \in \mathbb{R}^n,\\
(-\Delta - k^2 + q)u &= 0 \quad && x \in \mathbb{R}^n,\\
u &= u^i + u^s && \\
\lim_{r\to\infty} r^{\frac{n-1}{2}}(\partial_r - i\,k)u^s &= 0 \quad && r = |x|
\end{aligned}
$$

Fundamental solution: By Φ we denote the causal fundamental solution to the background equations, i.e. the unique solution to

$$
\begin{aligned}
(-\Delta - k^2)\Phi(x) &= \delta_0(x) \quad && x \in \mathbb{R}^n\\
\lim_{r\to\infty} r^{\frac{n-1}{2}}(\partial_r - i\,k)\Phi &= 0 \quad && r = |x|
\end{aligned}
$$

E.g. in 3D and higher dimensions $\Phi(x,k) = |x|^{-\frac{n-1}{2}}\exp(i\,k|x|)$. Note: for example in 3D, by passing to the time-domain, and setting $\phi(x,t) = \delta_0(t - |x|)\,/\,(4\pi|x|)$ we have $(\partial_t^2 - \Delta)\phi = \delta_0(t)\,\delta_0(x)$, the wave propagates to infinity, and moreover $\Phi(x,k) = \int_{-\infty}^{\infty}\phi(x,t)\exp(i\,t\,k)\,dt$.

Lippman–Schwinger equation: A numerically and function-theoretically useful way to solve the equation on the frequency domain is

$$
u(x) = u^i(x) - \int_{\mathbb{R}^n}\Phi(y-x)\,q(y)\,u(y)\,dy.
$$

Far-field pattern / scattering amplitude: These are the measurements of scattering experiments. One thinks of the incident wave u^i as an input, and of the far-field pattern u^s_∞ as output. We can show (in 3D) that $\Phi(y-x) = \exp(i\,k|x|)|x|^{-1}\,(\exp(-i\,k\,\hat{x}\cdot y) + \mathcal{O}(|x|^{-1}))$. Then

$$
\begin{aligned}
u^s(x) &= |x|^{-\frac{n-1}{2}}\exp(i\,k|x|)\,u^s_\infty(\hat{x}) + \mathcal{O}(|x|^{-n/2}),\\
u^s_\infty(\hat{x}) &= -\int_{\mathbb{R}^n}\exp(-i\,k\,\hat{x}\cdot y)\,q(y)\,u(y)\,dy.
\end{aligned}
$$

Born series / approximation: Build the total wave as successive approximations (or Neumann series) using the Lippmann–Schwinger equation. Let $L\,f(x) = -\int\Phi(y-x)\,q(y)\,f(y)\,dy$. Then the Born series is

$$
u = u^i + L\,u^i + L^2\,u^i + L^3\,u^i + \ldots
$$

and the Born approximation is $u \approx u^i + L u^i$ so the scattered field and far-field are approximated by

$$
\begin{aligned}
u^s(x) &\approx -\int_{\mathbb{R}^n}\Phi(y-x)\,q(y)\,u^i(y)\,dy,\\
u^s_\infty(x) &\approx -\int_{\mathbb{R}^n}\exp(-i\,k\,\hat{x}\cdot y)\,q(y)\,u^i(y)\,dy.
\end{aligned}
$$

Incident plane wave: Most of the inverse backscattering literature deals with incident plane waves (we will have initial point-source waves in the second part of the talk). Let $u^i(x) = \exp(i\, k\, x\cdot\theta)$ for a given $|\theta| = 1$. This is a plane wave propagating in the direction θ: $U^i(x,\, t) := \mathcal{F}_k\{u^i\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-i\, k\, t) u^i(x, k)\, dk = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-i\, k(\theta\cdot x - t))\, dk = \delta_0(\theta\cdot x - t)$ which indeed propagates along the vector θ as $t \to +\infty$. To emphasize the incident direction and frequency, from now on write

$$u_\infty^s(\hat{x}) = u_\infty^s(\hat{x}, \theta, k).$$

Inverse scattering problems: As a goal we want to recover q. We can probe for it by sending incident waves from admissible directions θ and measuring the far-field pattern at admissible directions $\hat{x}$.

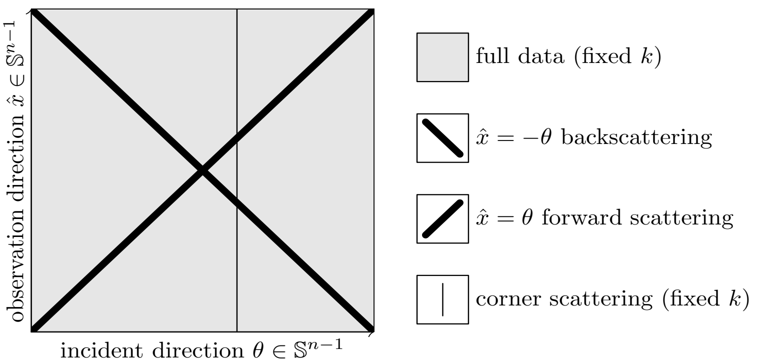

Figure 1.

Dimensions

Full data fixed freq:

$$\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \mapsto 2n-2$$

BS and FS:

$$\mathbb{S}^{n-1} \times \mathbb{R} \mapsto n$$

CS:

$$\mathbb{S}^{n-1} \mapsto n-1$$

(Some) past results of interest: This list is missing the Russian literature on the subject. I have heard that Novikov, Grinevich, Manakov and Kurylev among others have worked on this type of issues. Please send me references should you know more details.

Eskin & Ralston 1989. *The inverse backscattering problem in three dimensions.* Showed that the map $q \mapsto u_\infty^s(-\theta,\, \theta,\, k)$ is locally an analytic homeomorphism (bijection!) near any q in some particular set. Only know $q = 0$ is in this set and, dense + open?

Stefanov 1990. *A uniqueness result for the inverse backscattering problem.* Showed that

$$q_1 \geqslant q_2 \quad \text{and for some } \theta_0 \quad u_{1\infty}^s(-\theta_0, \theta_0, k) = u_{2\infty}^s(-\theta_0, \theta_0, k)\ \forall k \quad \text{then} \quad q_1 = q_2.$$

Päivärinta & Somersalo 1991. *Inversion of discontinuities for the Schrödinger equation in three dimensions.* Idea, given u_∞^s (more general data than BS), if q recovered not from the Lippmann-Schwinger equation but from the Born approximation, i.e. q_B then can we say something useful about q even when no smallness assumptions? Yes, $q - q_B$ smoother than q.

Greenleaf & Uhlmann 1993. *Recovering singularities of a potential from singularities of scattering data.* Time domain scattering with potential q cornomal distribution of low enough negative order

$$\begin{aligned} (\partial_t^2 - \Delta - q)U &= 0 \qquad x \in \mathbb{R}^3, \quad t \in \mathbb{R} \\ U(x,t) &= \delta_0(x\cdot\theta - t) \qquad x \in \mathbb{R}^3, \quad t \ll 0 \end{aligned}$$

Then principal symbol of q can be recovered from symbol of $U_\infty^s(-\theta, \theta, t)$, e.g. recover jumps.

Stefanov & Uhlmann 1997. *Inverse backscattering for the acoustic equation.* Time-domain backscattering, $(\partial_t^2 - c^2(x)\Delta)u = 0$, if $\|c(x) - 1\|_{W^{10,\infty}} < \varepsilon$ then uniqueness for c from BS data.

Ola, Päivärinta, Serov 2001. *Recovering singularities from backscattering in two dimensions.* Idea: with a plane-wave the Born approximation gives

$$u^s_\infty(-\theta,\theta,k) \approx u^B_\infty(\theta,k) = -\int_{\mathbb{R}^n} \exp(i\,k\,\theta\cdot y)\; q(y) \exp(i\,k\,y\cdot\theta)(y)\,dy = -\mathcal{F}^{-1}\{q\}(2k\theta)$$

so then define $B(\xi) = u^s_\infty(-\hat{\xi},\hat{\xi},|\xi|/2)$ and the Born approximated potential $q_B = -\mathcal{F}\{B\}$. Then the "principal singularities" of q can be recovered:

$$q \in H^{s_0} \quad \Longrightarrow \quad q - q_B \in H^{s_0+\varepsilon}$$

Ruiz & Vargas 2005. *Partial recovery of a potential from backscattering data.* Improve Ola–Päivärinta–Serov and do 3D also.

Reyes 2007. *Inverse backscattering for the Schrödinger equation in 2D.* Still improve Ola–Päivärinta–Serov, get $1/2$ derivative from Born approximation.

Stefanov & Uhlmann 2009. *Linearizing non-linear inverse problems and an application to inverse backscattering.* If linearization of map between Banach spaces is injective with closed range, then the original problem has local uniqueness and Lipschitz stability. As an example show Hölder stability for $(\partial_t^2 - c^2(x)\Delta)$ backscattering.

Rakesh & Uhlmann 2014. *Uniqueness for the inverse backscattering problem for angularly controlled potentials.* Time domain backscattering. If $q_1 - q_2$ angularly controlled + same backscattering data then they are equal.

Rakesh & Uhlmann 2015. *The point-source inverse backscattering problem.* Same as above but for the point-source problem (defined later in the talk).

Caro, Helin, Lassas 2016. *Inverse scattering for a random potential.* Determines the principal symbol of the covariance operator of a random potential from a single realization of the backscattering measurements.

Easy-looking open questions:

Almost everything is still open for non-singular potentials:

- $u^s_\infty(-\theta,\theta,k) = 0$ for all $k \in \mathbb{R}$ and $|\theta| = 1$, does this imply $q = 0$ if a-priori $q \in C_0^\infty(\mathbb{R}^n)$?
- other equations, e.g. Maxwell? Heat?

1D case: This is is more or less equivalent to the full data full frequency case in higher dimensions. Has been solved in the 60's and 70's. See Gel'fand–Levitan, Marchenko, and Gopinath–Sondhi.

2 Point-source backscattering

Problem statement: Given a potential q compactly supported in the unit disc B, for any source $a \in \partial B$ define the (time-domain) point-source problem

$$(\partial_t^2 - \Delta - q)U^a(x,t) = \delta_0(x-a)\,\delta_0(t) \qquad x \in \mathbb{R}^3, \quad t \in \mathbb{R}, \tag{1}$$

$$U^a(x,t) = 0 \qquad x \in \mathbb{R}^3, \quad t < 0. \tag{2}$$

If $U^a_1(a,t) = U^a_2(a,t)$ when $t > 0$ for two potentials q_1 and q_2, then do we have $q_1 = q_2$?

Angular control:

A function f defined in the unit disc B is angularly controlled if

$$\sum_{i<j} \int_{|x|=r} |\Omega_{ij} f(x)|^2 \, d\sigma(x) \leqslant S^2 \int_{|x|=r} |f(x)|^2 \, d\sigma(x)$$

for all $0<r<1$ where $\Omega_{ij} = x_i \partial_j - x_j \partial_i$ are the tangential vector fields at x on the sphere $|x|=r$.

Stability for point-source backscattering:

Theorem 1. *Let $q_1, q_2 \in C_c^7(B)$ with supports distance $h>0$ from ∂B. Then*

$$\|q_1 - q_2\|_{L^2(\{|x|=r\})} \leqslant e^{C/r^4} \|U_1^a - U_2^a\|_{\mathrm{BS}}$$

where

$$\|F\|_{\mathrm{BS}} = \sup_{0<\tau<1} \int_{a \in \partial B} |\partial_\tau(\tau F(a, 2\tau))|^2 \, d\sigma(a).$$

A fortiori if $\|U_1^a - U_2^a\|_{\mathrm{BS}} < \varepsilon$ *then*

$$\|q_1 - q_2\|_{L^2(B)} \leqslant C' \left(\ln \frac{1}{\|U_1^a - U_2^a\|} \right)^{-1/4}.$$

Well-posedness of direct problem:

This is "well known" if q infinitely smooth. However impossible to find sources with finite smoothness giving good enough estimates.

Theorem 2. *The above problem has a unique solution in the set of distributions of order ℓ when $q \in C_c^\ell(B)$. It is given by*

$$U^a(x,t) = \frac{\delta_0(t - |x-a|)}{4\pi |x-a|} + H(t - |x-a|)\, r^a(x,t) \tag{3}$$

and if $\ell \geqslant 7$ then $r^a \in C^1(\mathbb{R}^3 \times \mathbb{R})$ with locally finite norm bound. Moreover U^a is C^1 outside the characteristic cone $t = |x-a|$.

Contribution of my stability paper:

The Rakesh–Uhlmann-proof leads itself quite well for a stability estimate. However what was missing was Theorem 2, i.e. well-posedness with suitable norm estimates. Since this talk is about backscattering I will present the inverse problem solution instead. Moreover it is surprising that the final estimate is of logarithmic type. A-priori one would have guessed a Lipschitz or Hölder-type estimate since there is no exponential solutions involved.

Analogue to "Alessandrini-type identity":

Solving inverse problems always requires an identity tying the boundary measurements to the unknown potential. Here they are

$$(U_1^a - U_2^a)(a, 2\tau) = \frac{1}{32\pi^2 \tau^2} \int_{|x-a|=\tau} (q_1 - q_2)(x) \, d\sigma(x) + \int_{|x-a| \leqslant \tau} (q_1 - q_2)(x)\, k(x, \tau, a) \, dx \tag{4}$$

for $t>0$ where the kernel k is given by

$$k(x,\tau,a) = \frac{(r_1^a + r_2^a)(x, 2\tau - |x-a|)}{4\pi |x-a|} + \int_{|x-a|}^{2\tau - |x-a|} r_1^a(x, 2\tau - t)\, r_2^a(x,t) \, dt.$$

Under $r_1^a, r_2^a \in C^1$ we have $k \in C^1$ when $|x-a|>0$. That's why we require that $d(\operatorname{supp} q_j, \partial B) \geqslant h > 0$. How to prove the above? Calculate the following by using (1)–(2) first, and then (3):

$$\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} (q_1-q_2)(x)\, U_1^a(x,t)\, U_2^a(x, 2\tau - t)\, dx\, dt$$

Estimating the first term: This geometrically nontrivial step works for any $Q \in C_c^1(B)$, $|a|=1$ and $0<t<1$:

$$\partial_\tau\left(\frac{\tau}{4\pi\tau^2}\int_{|x-a|=\tau} Q(x)\, d\sigma(x)\right) = \frac{1-\tau}{2} Q((1-\tau)a) + E(a,\tau), \tag{5}$$

$$|E(a,\tau)|^2 \leqslant \frac{3}{\pi(1-\tau)}\sum_{i<j}\int_{|x-a|=\tau}\frac{|\Omega_{ij}Q(x)|}{\sqrt{|x|-(1-\tau)}}\, d\sigma(x). \tag{6}$$

Useful integral identities:

$$\begin{aligned}
\int_{|a|=1}\int_{|x-a|=\tau} f(x)\, d\sigma(x)\, d\sigma(a) &= 2\pi\tau\int_{|x|\geqslant 1-\tau}\frac{f(x)}{|x|}\, dx\\
\int_{|a|=1}\int_{|x-a|\leqslant\tau} f(x)\, d\sigma(x)\, d\sigma(a) &= \pi\int_{|x|\geqslant 1-\tau}\frac{f(x)}{|x|}(t^2-(1-|x|)^2)\, dx
\end{aligned}$$

Proof of stability of the inverse problem: Write $\delta U^a = U_1^a - U_2^a$ and $\delta q = q_1 - q_2$. Then start by multiplying by τ and differentiating the "Alessandrini-type" identity (4), and using (5).

$$\begin{aligned}
\partial_\tau(\tau\,\delta U^a(a,2\tau)) &= \frac{1-\tau}{16\pi}\delta q((1-\tau)a) + \frac{1}{8\pi}E(a,\tau) + \int_{|x-a|=\tau}\delta q(x)\,\tau\, k(x,\tau,a)\, d\sigma(x)\\
&\quad + \int_{|x-a|\leqslant\tau}\delta q(x)\,\partial_\tau(\tau k(x,\tau,a))\, dx.
\end{aligned}$$

Use the C^1-estimates for k and the estimate (6) to get

$$\begin{aligned}
(1-\tau)^2|\delta q((1-\tau)a)|^2 &\lesssim |\partial_\tau(\tau\,\delta U^a(a,2\tau))|^2 + (1-\tau)^{-1}\sum_{i<j}\int_{|x-a|=\tau}\frac{|\Omega_{ij}\delta q(x)|}{\sqrt{|x|-(1-\tau)}}\, d\sigma(x)\\
&\quad + \int_{|x-a|=\tau}|\delta q(x)|^2\, d\sigma(x) + \int_{|x-a|\leqslant\tau}|\delta q(x)|^2\, d\sigma(x).
\end{aligned}$$

Then integrate over $|a|=1$ and use the useful integral identities

$$\begin{aligned}
\int_{|x|=1-\tau}|\delta q(x)|^2\, d\sigma(x) &\lesssim \int_{|a|=1}|\partial_\tau(\tau\,\delta U^a(a,2\tau))|^2\, d\sigma(a) + \frac{\tau}{1-\tau}\sum_{i<j}\int_{|x|\geqslant 1-\tau}\frac{|\Omega_{ij}\delta q(x)|\, d\sigma(x)}{|x|\sqrt{|x|-(1-\tau)}}\\
&\quad + \int_{|x|\geqslant 1-\tau}|\delta q(x)|^2\frac{\tau^2+2\tau-(1-|x|)^2}{|x|}\, dx.
\end{aligned}$$

Simple algebra, the assumption of angular control for δq and having $1-\tau \geqslant \varepsilon > 0$ gives

$$\int_{|x|=1-\tau}|\delta q(x)|^2\, d\sigma(x) \lesssim \|\delta U^a\|_{\mathrm{BS}} + C\varepsilon^{-2}\int_0^\tau\frac{1}{\sqrt{\tau-s}}\int_{|x|=1-s}|\delta q(x)|^2\, d\sigma(x)\, ds.$$

Applying Grönwall's inequality ($\varphi(\tau) \leqslant C_1 + C_2\int_0^\tau \varphi(s')\, ds' \Rightarrow \varphi(\tau) \leqslant C_1 \exp(C_2\tau)$) gives the claim.

$$\|q_1-q_2\|_{L^2(\{|x|=r\})} \leqslant e^{C/r^4}\|U_1^a - U_2^a\|_{\mathrm{BS}}$$

and also, if $\|U_1^a - U_2^a\|_{\mathrm{BS}} < \varepsilon$ then

$$\|q_1-q_2\|_{L^2(B)} \leqslant C'\left(\ln\frac{1}{\|U_1^a-U_2^a\|}\right)^{-1/4}.$$