Transmission eigenfunction localization
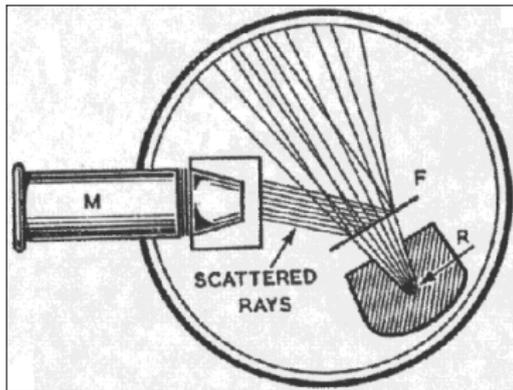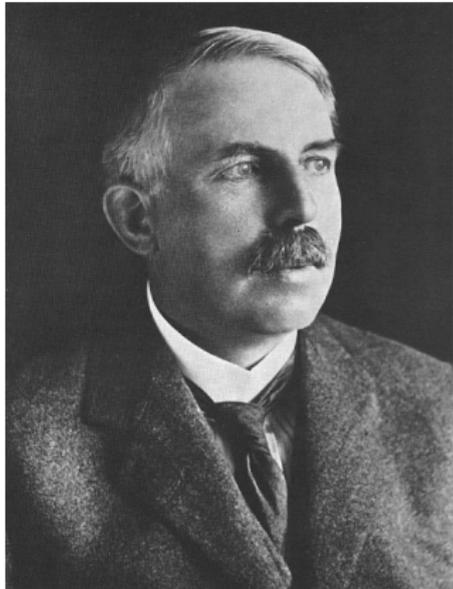
Eemeli Blåsten

Institute for Advanced Study,
The Hong Kong University of Science and Technology

Annual meeting of the Hong Kong Mathematical Society
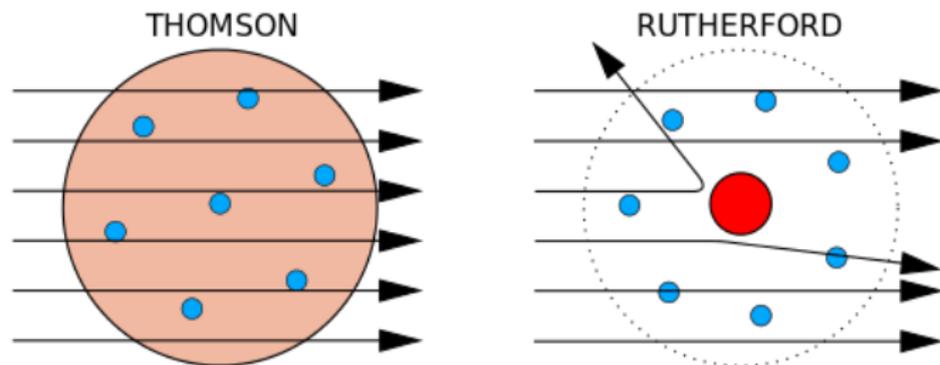
HKUST, May 20, 2017
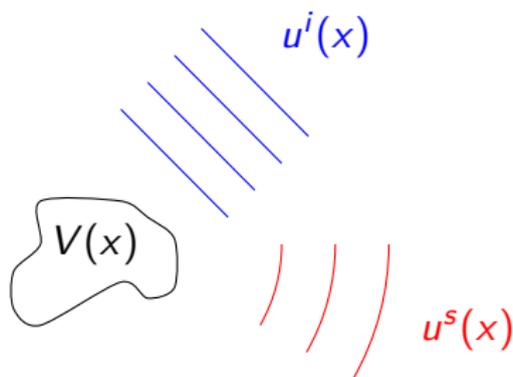
Scattering theory
Lord Rutherford's gold-foil experiment

Scattering theory
Rutherford experiment's conclusions

measurement + a-priori information = conclusion

Scattering theory

The total wave u satisfies

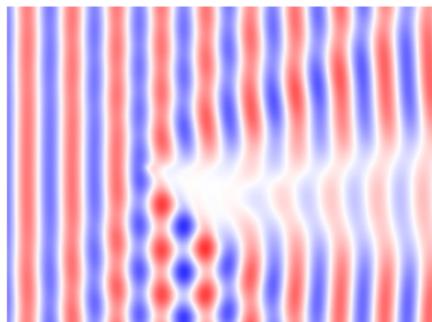$$(\Delta + k^2(1 + V))u = 0,$$

V models a perturbation of the background,

$$u = u^i(x) + u^s(x)$$

incident wave   scattered wave

Scattering theory
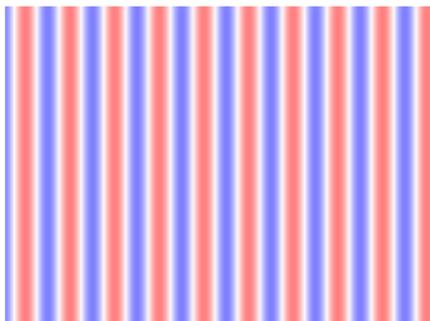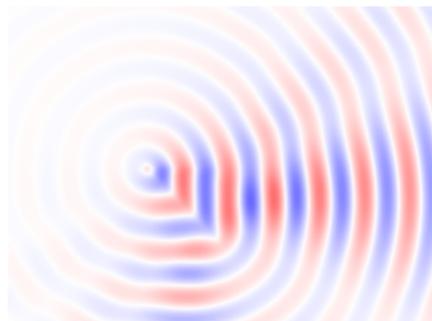
Mathematical scattering theory: measurements
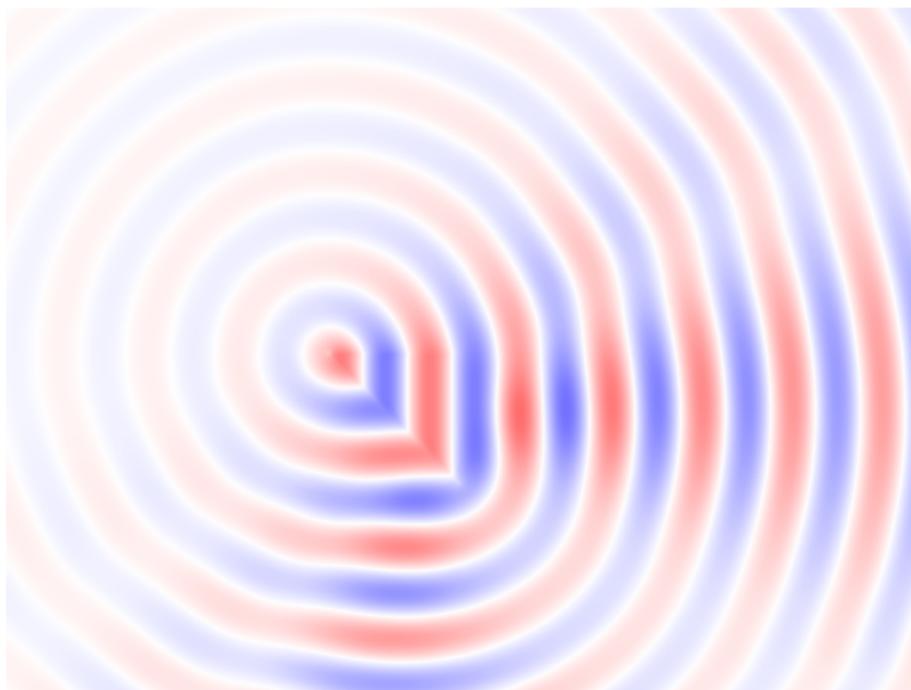
Measurement: A_{u^i} is the far-field pattern of the scattered wave

$$u^s(x) = \frac{e^{ik|x|}}{|x|^{(n-1)/2}} A_{u^i}\left(\frac{x}{|x|}\right) + \mathcal{O}\left(\frac{1}{|x|^{n/2}}\right)$$

Inverse problem

Given the map $u^i \mapsto A_{u^i}$, recover V or its support Ω.

Early methods (< 85')
  - optimization and minimization methods

Inverse problem

Given the map $u^i \mapsto A_{u^i}$, recover V or its support Ω.

Early methods ($< 85'$)

- optimization and minimization methods

Sampling methods

- gives condition on measurements for $x \in \operatorname{supp} V$
- compared to before: *fast! works reliably!*
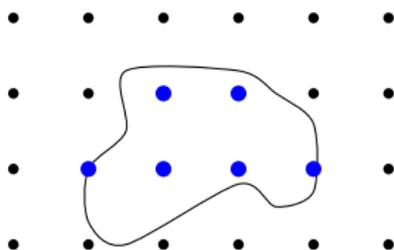
Everything solved?

Everything solved?

NO!

If $A_{u^i} = 0$ for some $u^i \neq 0$ then the above methods fail!

Everything solved?

NO!

If $A_{u^i} = 0$ for some $u^i \neq 0$ then the above methods fail!

$A_{u^i} = 0$ implies

$$(\Delta + k^2)u^i = 0, \qquad \Omega$$
$$(\Delta + k^2(1 + V))u = 0, \qquad \Omega$$
$$u - u^i \in H_0^2(\Omega)$$

k^2 is an interior transmission eigenvalue (ITE)

Some ITE history

- 86', 88' **Kirsch, Colton–Monk**: ITE problem posed
- 89', 91' **Colton–Kirsch–Päivärinta, Rynne–Sleeman**: discreteness of ITE

Some ITE history

- 86', 88' **Kirsch, Colton–Monk**: ITE problem posed
- 89', 91' **Colton–Kirsch–Päivärinta, Rynne–Sleeman**: discreteness of ITE
- 91'–08' NOTHING. . .

Some ITE history

- 86', 88' **Kirsch, Colton–Monk**: ITE problem posed
- 89', 91' **Colton–Kirsch–Päivärinta, Rynne–Sleeman**: discreteness of ITE
- 91'–08' NOTHING. . .
- 07', 09' **Cakoni–Colton–Monk, Cakoni–Colton–Haddar**: qualitative information about V from ITE's
- 08' **Päivärinta–Sylvester**: existence for general scatterers
- 10' **Cakoni–Gintides–Haddar**: infinitely many ITE's
- 10' **Cakoni–Colton–Haddar**: ITE's can be deduced from far-field data
- 10'+: EXPLOSION OF INTEREST

Some ITE history

- 86', 88' **Kirsch, Colton–Monk**: ITE problem posed
- 89', 91' **Colton–Kirsch–Päivärinta, Rynne–Sleeman**: discreteness of ITE
- 91'–08' NOTHING...
- 07', 09' **Cakoni–Colton–Monk, Cakoni–Colton–Haddar**: qualitative information about V from ITE's
- 08' **Päivärinta–Sylvester**: existence for general scatterers
- 10' **Cakoni–Gintides–Haddar**: infinitely many ITE's
- 10' **Cakoni–Colton–Haddar**: ITE's can be deduced from far-field data
- 10'+: EXPLOSION OF INTEREST
- interest shifting to "Steklov eigenvalues"

Corner scattering

- B.–Päivärinta–Sylvester 14: $V = \chi_{[0,\infty[^n}\varphi$, $\varphi(0) \neq 0$ always scatters, *despite having interior transmission eigenvalues*

Corner scattering

- B.–Päivärinta–Sylvester 14: $V = \chi_{[0,\infty[^n}\varphi$, $\varphi(0) \neq 0$ always scatters, *despite having interior transmission eigenvalues*

$$k^2 \text{ ITE } and \ A_{u^i} \neq 0 \ \forall u^i \neq 0$$

Proof sketch

Rellich's theorem and unique continuation imply

$$k^2 \int V u^i u_0 dx = 0$$

if $(\Delta + k^2(1 + V))u_0 = 0$ near $\operatorname{supp} V$.

Proof sketch

Rellich's theorem and unique continuation imply

$$k^2 \int V u^i u_0 \, dx = 0$$

if $(\Delta + k^2(1 + V))u_0 = 0$ near $\operatorname{supp} V$.
In simple case

$$
\begin{aligned}
u^i(x) &= u^i(0) + u_r^i(x) \\
u_0(x) &= e^{\rho \cdot x}(1 + \psi(x)) \\
V(x) &= \chi_{[0,\infty[^n}(x)(\varphi(0) + \varphi_r(x))
\end{aligned}
$$

Proof sketch

Rellich's theorem and unique continuation imply

$$k^2 \int V u^i u_0 \, dx = 0$$

if $(\Delta + k^2(1 + V))u_0 = 0$ near $\operatorname{supp} V$.
In simple case

$$
\begin{aligned}
u^i(x) &= u^i(0) + u_r^i(x) \\
u_0(x) &= e^{\rho \cdot x}(1 + \psi(x)) \\
V(x) &= \chi_{[0,\infty[^n}(x)(\varphi(0) + \varphi_r(x))
\end{aligned}
$$

Hölder estimates give

$$
C \left| \varphi(0) u^i(0) \right| |\rho|^{-n} \le \left| \varphi(0) u^i(0) \int_{[0,\infty[^n} e^{\rho \cdot x} \, dx \right| \le C |\rho|^{-n-\delta}
$$

if $\|\psi\|_p \le C |\rho|^{-n/p-\varepsilon}$.

Newer corner scattering results

- Päivärinta–Salo–Vesalainen: 2D any angle, 3D almost any spherical cone
- Hu–Salo–Vesalainen: smoothness reduction, new arguments, *polygonal scatterer probing*
- Elschner–Hu: 3D any domain having two faces meet at an angle

Newer corner scattering results

- Päivärinta–Salo–Vesalainen: 2D any angle, 3D almost any spherical cone
- Hu–Salo–Vesalainen: smoothness reduction, new arguments, *polygonal scatterer probing*
- Elschner–Hu: 3D any domain having two faces meet at an angle

Injectivity of support probing:

Theorem
Let P, P' be convex polygons and $V = \chi_P \varphi$, $V' = \chi_{P'} \varphi'$ for admissible functions φ, φ'. Then

$$P \neq P' \quad \implies \quad A_{u^i} \neq A'_{u^i} \quad \forall u^i \neq 0$$

Any *single* incident wave determines P in the class of polygonal penetrable scatterers.

More recent work: lower bound for far-field pattern

Theorem (B., Liu, preprint)

Let u^i be a normalized Herglotz wave,

$$u^i(x) = \int_{\mathbb{S}^{n-1}} e^{ik\theta \cdot x} g(\theta) d\sigma(\theta), \qquad \|g\|_{L^2(\mathbb{S}^{n-1})} = 1,$$

and let $V = \chi_P \varphi$ be admissible.

More recent work: lower bound for far-field pattern

Theorem (B., Liu, preprint)

Let u^i be a normalized Herglotz wave,

$$u^i(x) = \int_{\mathbb{S}^{n-1}} e^{ik\theta \cdot x} g(\theta) d\sigma(\theta), \qquad \|g\|_{L^2(\mathbb{S}^{n-1})} = 1,$$

and let $V = \chi_P \varphi$ be admissible. Then

$$\|A_{u^i}\|_{L^2(\mathbb{S}^{n-1})} \geq C_{\|P_N\|, V} > 0$$

where the Taylor expansion of u^i at the corner x_c begins with P_N, and $\|P_N\| = \int_{\mathbb{S}^{n-1}} |P_N(\theta)| \, d\sigma(\theta)$.

Mistake?

F. Cakoni: "Incident waves that approximate transmission eigenfunctions produce arbitrarily small far-field patterns."

Mistake?

F. Cakoni: "Incident waves that approximate transmission eigenfunctions produce arbitrarily small far-field patterns."

Well-known: Herglotz waves can approximate transmission eigenfunctions.

Mistake?

F. Cakoni: "Incident waves that approximate transmission eigenfunctions produce arbitrarily small far-field patterns."

Well-known: Herglotz waves can approximate transmission eigenfunctions.

So, mistake in our proof?

Mistake?

F. Cakoni: "Incident waves that approximate transmission eigenfunctions produce arbitrarily small far-field patterns."

Well-known: Herglotz waves can approximate transmission eigenfunctions.

So, mistake in our proof?
– No: $C = C_{\|P_N\|}$, so the bound becomes arbitrarily small for incident waves that have small value at the corner.

From contradiction to inspiration

Theorem (B., Liu, preprint)

Let the potential $V = \chi_P \varphi$ be admissible and $P \subset \Omega$. Let v be a transmission eigenfunction

$$(\Delta + k^2)v = 0, \qquad \Omega$$
$$(\Delta + k^2(1 + V))w = 0, \qquad \Omega$$
$$w - v \in H_0^2(\Omega), \qquad \|v\|_{L^2(\Omega)} = 1.$$

From contradiction to inspiration

Theorem (B., Liu, preprint)

Let the potential $V = \chi_P \varphi$ be admissible and $P \subset \Omega$. Let v be a transmission eigenfunction

$$
\begin{aligned}
(\Delta + k^2)v &= 0, &\quad \Omega \\
(\Delta + k^2(1 + V))w &= 0, &\quad \Omega \\
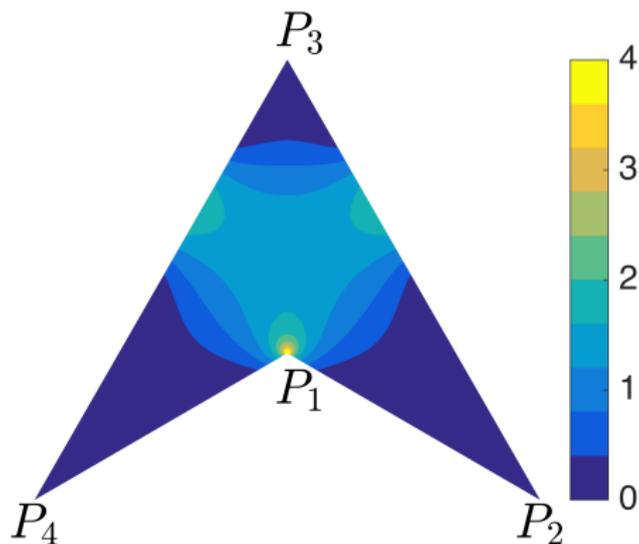w - v &\in H_0^2(\Omega), &\quad \|v\|_{L^2(\Omega)} = 1.
\end{aligned}
$$

If v can be approximated by a sequence of Herglotz waves with *uniformly L^2-bounded kernels g*, then

$$
\lim_{r \to 0} \frac{1}{m(B(x_c, r))} \int_{B(x_c, r)} |v(x)| \, dx = 0
$$

at every corner point x_c of $\operatorname{supp} V$.

Transmission eigenfunction localization

Numerical investigation with Xiaofei Li, Hongyu Liu and Yuliang Wang:

Thank you for your attention!