Non-scattering energies, new resolvent estimates and other projects

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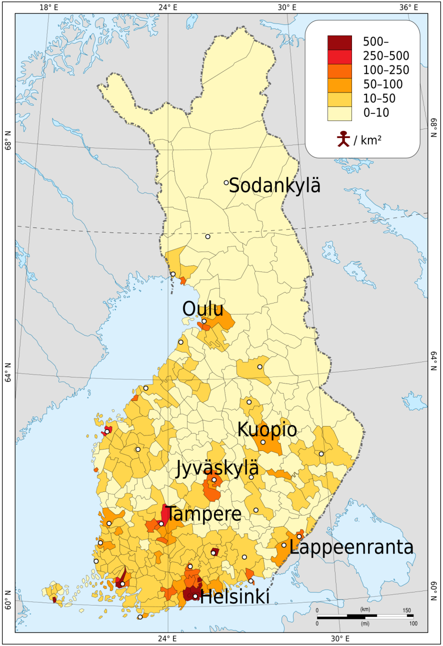

18° E
24° E
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500–
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Sodankylä
Oulu
Kuopio
Jyväskylä
Tampere
Lappeenranta
Helsinki
0
(km)
150
(mi)
100

Projects in chronological order

Published work:

- 2D inverse scat
- transmission eigenvalues
- non-scattering energies, + hyperbolic

Work in progress:

- new estimates
- scattering solutions for general PDE?
- building on Rakesh–Uhlmann-type backscattering
- 1D water pipe networks
- LaTeX on the web

Inverse scattering problem in 2D

with
Oleg Imanuvilov (Colorado State University),
Masahiro Yamamoto (University of Tokyo),
Yang Yang (Purdue University)

Inverse problems for partial differential equations

The Calderón problem: given an open set $\Omega \subset \mathbb{R}^n$ and all (voltage, current flux) pairs $(v, f) \in H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ satisfying

$$\begin{aligned} \nabla \cdot \gamma \nabla u = 0 &\quad \Omega \\ u = v &\quad \partial\Omega \\ \gamma \partial_\nu u = f &\quad \partial\Omega \end{aligned}$$

deduce the conductivity γ inside Ω.

- statement + linearized problem: Calderón (60's / 1980)
- 3D isotropic: Sylvester & Uhlmann (1987)
- 2D isotropic: Astala & Päivärinta (2006)

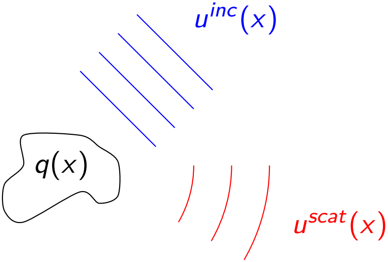
Inverse potential scattering

Inverse scattering: given

$$\mathcal{C}_q = \left\{ (u_{|\partial\Omega}, \partial_\nu u_{|\partial\Omega}) \,\middle|\, (\Delta + q)u = 0, u \in H^1(\Omega) \right\},$$

deduce the scattering potential q inside Ω.

With potential reaching infinity: given the scattering matrix $S_q(\lambda)$ for a fixed frequency or wavenumber λ, determine q.



$$(\Delta + q + \lambda^2)(u^{inc} + u^{scat}) = 0$$

$S_q(\lambda)$ relates the behaviours of u^{inc} and u^{scat} at infinity.

Important papers (smoothness p.o.v.)

- Calderón 1980 (manuscript from 60's): linearised problem
- Kohn–Vogelius 1984: piecewise analytic γ
- Sylvester–Uhlmann 1987: C^k γ and q in 3D
- Alessandrini 1988: logarithmic stability result
- Astala–Päivärinta 2006: arbitrary $\gamma \in L^\infty(\Omega)$ in 2D
- Bukhgeim 2008: 2D uniqueness $q \in W^{1,p}(\Omega)$
- Novikov–Santacesaria 2010: stability for $q \in C^2(\Omega)$ in 2D
- B.–Imanuvilov–Yamamoto's contributions: uniqueness $q \in L^p$, stability $q \in W^{\varepsilon,p}$ in 2D
- Haberman–Tataru 2013: 3D uniqueness, $\gamma \in C^1$
- Caro–Rogers 2015: 3D uniqueness, γ Lipschitz

Partial data results avoided in this list!
Also, among others: Nachman, Liu, Jerison, Kenig, ...

Typical way of solving 2D potential scattering inverse problems

If q_1 and q_2 give the same measurement results, then

$$\int (q_1 - q_2) u_1 u_2 dm = 0$$

for all admissible u_j satisfying

$$(\Delta + q_j) u_j = 0.$$

Complex Geometric Optics solutions in 2D (Bukhgeim 2008)

$$u(z) = e^{i\tau\Phi(z)}(1 + \varepsilon(z)), \quad \overline{\partial}\Phi = 0.$$

Stationary phase method (if e.g. $\Phi(z) = z^2$)

$$\lim_{\tau\to\infty} \frac{2\tau}{\pi} \int_{\mathbb{C}} e^{i\tau(\Phi+\overline{\Phi})}(q_1 - q_2)(z) dm(z) = (q_1 - q_2)(z_0)$$

Contributions by myself and collaborators

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and $p > 2$.

Theorem (Imanuvilov–Yamamoto 2012)

Assume that $q_1, q_2 \in L^p(\Omega)$ with $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$. Then $q_1 = q_2$.

Theorem (B. 2013)

Let $\varepsilon > 0$ and $M < \infty$. Then there exists constants $C, d_0, \theta > 0$ such that

$$\|q_1 - q_2\|_{L^2(\Omega)} \leq C \left(\ln \frac{1}{d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})} \right)^{-\theta}$$

if $q_1, q_2 \in W_p^\varepsilon(\Omega)$ with norms at most M and $d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) \leq d_0$.

Conjecture (B.–Yang)

Assume that $q_1, q_2 \in L^{(2,1)}(\mathbb{R}^2) \cap e^{-c|z|^2} L^1(\mathbb{R}^2) \quad \forall c > 0$ with $S_{q_1}(0) = S_{q_2}(0)$. Then $q_1 = q_2$.

What is the scattering matrix at 0 energy?!

Interior transmission eigenvalues

with

Lassi Päivärinta (Tallinn University of Technology)

The interior transmission problem

The interior transmission problem (of the Helmholtz equation) for the potential V is the following boundary value problem:

$$\begin{aligned} (\Delta - \lambda)v = 0 \quad &\text{in} \quad \Omega, \\ (\Delta - \lambda(1 + V))w = 0 \quad &\text{in} \quad \Omega, \\ v - w \in H_0^2(\Omega). \end{aligned} \tag{ITP}$$

We say that $\lambda \in \mathbb{C}$ is a transmission eigenvalue (TE) if (ITP) has non-trivial solutions $0 \neq v \in L^2_{\mathrm{loc}}$ and $0 \neq w \in L^2_{\mathrm{loc}}$.

Interior Transmission Problem

Why interesting:

- generalized eigenvalue problem (analytic Fredholm theory)
- resonant frequencies for *penetrable* scatterers
- ITE's show up in the far field data
- can V be determined from ITP spectrum?

Some history:

- 86', 88' **Kirsch, Colton–Monk**: ITP posed
- 89', 91' **Colton–Kirsch–Päivärinta, Rynne–Sleeman**: discreteness of ITE
- 91'–08' NOTHING...
- 07', 09' **Cakoni–Colton–Monk, Cakoni–Colton–Haddar**: qualitative information about V from ITE's
- 08' **Päivärinta–Sylvester**: existence for general scatterers
- 10' **Cakoni–Gintides–Haddar**: infinitely many ITE's
- 10' **Cakoni–Colton–Haddar**: ITE's can be deduced from far-field data
- 10'+: EXPLOSION OF INTEREST

Our goal was

Cakoni, Gintides, Haddar 2010: "We think that some interesting open problems [are] . . . , . . . and the completeness of the eigensystem of the interior transmission problem."

With Päivärinta we proved the completeness of a system of generalized eigenstates (2013).

Characterization of TE's

$$
\begin{aligned}
(\Delta - \lambda)v = 0 \quad \text{in} \quad \Omega, \\
(\Delta - \lambda(1+V))w = 0 \quad \text{in} \quad \Omega, \\
v - w \in H_0^2(\Omega).
\end{aligned}
$$

implies for $u = v - w$

$$
\begin{aligned}
u \in H^4 \cap H_0^2(\Omega), \\
T(\lambda)u := (\Delta - \lambda(1+V))\frac{1}{V}(\Delta - \lambda)u = 0.
\end{aligned}
$$

Reduction to a higher-order eigenvalue problem

Under some conditions

$$0 \neq \lambda \in \mathbb{C} \text{ is a TE}$$

$$\Longleftrightarrow$$

there is $0 \neq u \in H_0^2(\Omega)$ solving the following quadratic eigenvalue problem

$$T(\lambda)u = (A_0 + A_1\lambda + A_2\lambda^2)u = 0,$$

$$A_0 = \Delta \frac{1}{V} \Delta, \quad A_1 = -\frac{1}{V}\Delta - \Delta\frac{1}{V} - \Delta, \quad A_2 = 1 + \frac{1}{V}.$$

What are generalized transmission eigenstates?

Keywords: root vectors, chain of associated elements, Keldysh

Let λ_0 be a TE and B_0, B_1 and B_2 the Taylor coefficients centered at λ_0:

$$T(\lambda) = B_0 + B_1(\lambda - \lambda_0) + B_2(\lambda - \lambda_0)^2.$$

Definition

The generalized eigenspace $\mathcal{E}_{\lambda_0}$ is the closed linear space spanned by the vectors $(u_j)_{j=0}^{\infty}$, $u_j \in H_0^2(\Omega)$, where

$$\begin{aligned}
&B_0 u_0 = 0, \qquad u_0 \neq 0,\\
&B_1 u_0 + B_0 u_1 = 0,\\
&B_2 u_{j-2} + B_1 u_{j-1} + B_0 u_j = 0, \quad j = 2, 3, \ldots
\end{aligned}$$

An easier definition

A vector w is a generalized eigenvector of a matrix M if

$$(M - \lambda_0)^k w = 0$$

for some eigenvalue λ_0 and $k \in \mathbb{N}$.

Remark

Let $T(\lambda) = A_0 + A_1\lambda + A_2\lambda^2$. Then u_j, $j = 0, 1, \ldots$ are generalized eigenfunctions iff there is v_j such that

$$(\mathcal{A} - \lambda_0)^{j+1} \begin{pmatrix} u_j \\ v_j \end{pmatrix} := 0, \quad \textit{where } \mathcal{A} = \begin{pmatrix} 0 & A_2^{-1} \\ -A_0 & -A_1 A_2^{-1} \end{pmatrix}.$$

for $j = 0, 1, \ldots$

Completeness result

Theorem (B.–Päivärinta, 2013)

Assume that $V \in C^\infty(\overline{\Omega})$ and $V > 0$ on $\overline{\Omega}$. Then the space $\bigoplus_{\lambda \in \mathbb{C}} \mathcal{E}_\lambda$ is complete in $L^2(\Omega)$.

Tools:

- generalized Shapiro-Lopatinsky conditions by Agranovich and Vishik (1964) to invert $T(\lambda)$
- Nevanlinna theory to estimate $\|T(\lambda)^{-1}\|$
- the analytic Fredholm theorem

Names:

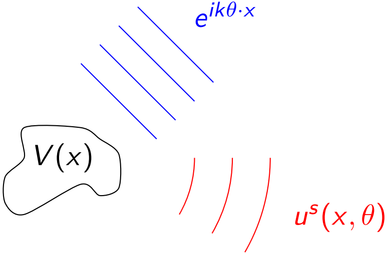
- Keldysh, Agranovich, Robert and Lai, Robbiano

Non-scattering energies

with

Valter Pohjola (Uppsala University),
Lassi Päivärinta (Tallinn University of Technology),
John Sylvester (University of Washington),
Esa Vesalainen (Aalto University)

Single frequency plane-wave scattering



The total wave u satisfies

$$\big(\Delta + k^2(1+V)\big)u = 0,$$

where V models a perturbation to the background wave speed and

$$u = \underbrace{e^{ik\theta\cdot x}}_{\text{incident wave}} + \underbrace{u^s(x,\theta)}_{\text{scattered wave}}$$

More general and realistic incident waves

$$u_g(x) = u_g^i(x) + \frac{e^{ik|x|}}{|x|} A_g\left(\frac{x}{|x|}\right) + O\left(\frac{1}{|x|^2}\right),$$

where

$$u_g^i(x) = \int_{\mathbb{S}^2} e^{ikx\cdot\theta} g(\theta) d\sigma(\theta)$$

is a superposition of plane-waves and A_g is the scattering amplitude of the scattered wave.

Vanishing Scattering Amplitude?

Question: Can there be $g \in L^2(\mathbb{S}^{n-1})$, $g \neq 0$ such that $A_g \equiv 0$?

Consequence:

$$\text{Rellich's theorem} \quad \Longrightarrow \quad u_g^s \equiv 0 \quad \mathbb{R}^n \setminus \operatorname{supp} V.$$

Recall that

$$\underbrace{u_g}_{=:\,w} \;=\; \underbrace{u_g^i}_{=:\,v} \;+\; u_g^s$$

compact support

Now v and w satisfy

$$\begin{aligned} (\Delta + k^2(1+V))w &= 0 \quad \text{in } \Omega & w &= v & &\text{on } \partial\Omega \\ (\Delta + k^2)v &= 0 \quad \text{in } \Omega & \tfrac{\partial}{\partial \nu} w &= \tfrac{\partial}{\partial \nu} v & &\text{on } \partial\Omega \end{aligned}$$

This is the interior transmission problem.

Non Scattering Energies

Definition

$\lambda > 0$ is a non-scattering energy (NSE) if the scattering amplitude is not injective, i.e. there is an incident wave

$$u_g^i(x) = \int_{\mathbb{S}^{n-1}} e^{ik\theta\cdot x} g(\theta) d\sigma(\theta), \quad k = \sqrt{\lambda},$$

such that the scattered wave u_s has zero far field, so $A_g \equiv 0$.

Remark

Non-scattering energies are transmission eigenvalues.

Theorem

$\{$*Transmission eigenvalues*$\} \neq \{$*non-scattering energies*$\}$
(B.–Päivärinta–Sylvester 2014, Päivärinta–Salo–Vesalainen submitted 2014)

Theorem and Consequences

Theorem (B.–Päivärinta–Sylvester, 2014)

Let $V = \chi_C \varphi$, $C =]0, \infty[^n$ *and* φ *with bounded support and* $\varphi(\overline{0}) \neq 0$. *Then* V *scatters all incident waves at all energies.*

Remark

Two penetrable scatterers whose difference is non-scattering at energy λ *can be distinguished from a single scattering measurement!*

Theorem (Hu–Salo–Vesalainen, 2016)

The shape of polygonal penetrable scatterers can be deduced from any single measurement in 2D. The same is true for rectangular scatterers in 3D.

Results used in the proof

Proposition

If v is a non-scattering incident wave, then

$$\int Vvw dx = 0$$

for all $w \in L^1_{loc}$, $(\Delta + k^2(1+V))w = 0$.

Proposition

Let H be a harmonic polynomial. If $\int_{x\geq 0} e^{\rho\cdot x} H(x)dx = 0$ for all $\rho \in \mathbb{C}^n$, $\rho\cdot\rho = 0$, $\Re\rho < 0$, then $H \equiv 0$.

Proposition

There is $w \in L^p_{loc}$, $2 \leq p < \infty$, $(\Delta + k^2(1+V))w = 0$, such that

$$w(x) = e^{\rho\cdot x}(1+\psi(x)), \qquad \|\psi\|_{L^p(\Omega)} \leq C_\Omega \left|\Im\rho\right|^{-1}.$$

Related things

Theorem (B.–Pohjola–Vesalainen, almost submitted)

In the hyperbolic space $\mathbb{H}^n$ hyperbolic rectangular ($n \in \mathbb{N}$) or spherical ($n \in \{2, 3\}$) penetrable cones always scatter.

Conjecture

Let M be a symmetric positive definite matrix. Let H be a polynomial and $P(D)H = \nabla \cdot (M \nabla H) = 0$. If

$$\int_{x \geq 0} e^{\rho \cdot x} H(x) dx = 0$$

for all $\rho \in \mathbb{C}^n$, $P(\rho) = 0$, $\Re \rho < 0$, then $H \equiv 0$.

Corollary

Polygonal scatterers always scatter also in 3D and higher.

New estimates for general PDE's

with

John Sylvester (University of Washington)

work in progress

New estimates for direct scattering theory

Old well-known estimates

Let $(\Delta + k^2)u = f$. then

- Agmon (1975), $\delta > \frac{1}{2}$

$$\left\|(1+|x|^2)^{-\delta/2}u\right\|_{L^2(\mathbb{R}^n)} \leq \frac{C}{k}\left\|(1+|x|^2)^{\delta/2}f\right\|_{L^2(\mathbb{R}^n)}$$

- Agmon–Hörmander (1976) $A_j = \{2^{j-1} < |x| < 2^{j+1}\}$, $A_0 = \{|x| < 2\}$.

$$\sup_{j\geq 0} \sqrt{2^j}^{-1} \|u\|_{L^2(A_j)} \leq \frac{C}{k}\sum_{j=0}^{\infty} \sqrt{2^j}\,\|f\|_{L^2(A_j)}$$

- Kenig–Ruiz–Sogge (1987) $\frac{1}{q_1} + \frac{1}{q_2} = 1$, $\frac{2}{n+1} \leq \frac{1}{q_1} - \frac{1}{q_2} \leq \frac{2}{n}$

$$\|u\|_{L^{q_2}(\mathbb{R}^n)} \leq Ck^{n(\frac{1}{q_1}-\frac{1}{q_2})-2}\|f\|_{L^{q_1}(\mathbb{R}^n)}$$

All of the above not satisfactory from a physical point of view: dilation, rotation, translation, behaviour w.r.t wavelength. . .

New estimates for direct scattering theory

Theorem (Sylvester 2013 or earlier)

If $\operatorname{supp} f \subset \Omega_s$ *then* $(\Delta + k^2)u = f$ *has a scattering solution* u*. It satisfies*

$$\|u\|_{L^2(\Omega_r)} \leq C\frac{\sqrt{\operatorname{diam}(\Omega_r)}\sqrt{\operatorname{diam}(\Omega_s)}}{k}\,\|f\|_{L^2(\Omega_s)}$$

for any bounded Ω_r*.*

Corollary

Agmon-Hörmander estimates follow.

Goal: Same estimate for $P(D)u = f$, P constant coefficient, $P^{-1}(0)$ non-singular.
Difficulty: Generality of P. Meta-inverse problem!

New CGO-estimate

Fundamental estimate, Sylvester–Uhlmann 1987, if $\rho \cdot \rho = 0$ then

$$\|(\Delta - 2\rho \cdot \nabla)^{-1} f\|_{L^2(\Omega)} \leq \frac{C}{|\rho|} \|f\|_{L^2(\Omega)}$$

Theorem (B.–Sylvester, to be published)

If $\frac{1}{p_2} - \frac{1}{p_1} < \frac{1}{n-1}$ *and* $p_2 \leq p_1$

$$\|(\Delta - 2\rho \cdot \nabla)^{-1} f\|_{L^\infty(\Re\rho, \widehat{L^{p_2}}(\Re\rho^\perp))} \leq C\, |\rho|^{(n-1)(\frac{1}{p_2} - \frac{1}{p_1}) - 1} \|f\|_{L^1(\Re\rho, \widehat{L^{p_1}}(\Re\rho^\perp))}$$

Corollary

If $\operatorname{supp} f \subset \Omega_s$, $\frac{1}{q_1} - \frac{1}{q_2} < \frac{1}{n-1}$ *and* $q_1 \leq 2 \leq q_2$ *then*

$$\|(\Delta - 2\rho \cdot \nabla)^{-1} f\|_{L^{q_2}(\Omega_w)} \leq C d(\Omega_w)^{1/q_2} d(\Omega_s)^{1/q_1} |\rho|^{(n-1)(\frac{1}{q_1} - \frac{1}{q_2}) - 1} \|f\|_{L^{q_1}(\Omega_s)}.$$

New estimates for direct scattering theory

Idea of the proof in 1D

$$(\partial_x^2 + k^2)u = f \implies (-\xi^2 + k^2)\hat{u} = \hat{f}$$

$$\hat{u} = -\frac{\hat{f}}{\xi^2 - k^2} = -\frac{\hat{f}}{2k}\left(\frac{1}{\xi - k} - \frac{1}{\xi + k}\right)$$

$$\hat{u} \text{ ``scattered''} := -\frac{\hat{f}}{2k}\left(\frac{1}{\xi - (k - i0)} - \frac{1}{\xi + (k - i0)}\right)$$

$$\mathscr{F}\{\sqrt{2\pi} i\, H(x)\, e^{izx}\}(\xi) = \frac{1}{\xi - z}, \quad \text{if } \operatorname{Im} z > 0.$$

Result

$$u \text{ ``scattered''} = f * \frac{ie^{-ik|x|}}{2k}, \qquad \|u\|_{L^\infty} \leq \frac{1}{2k}\|f\|_{L^1}$$

New estimates for direct scattering theory

Idea of the proof in 2D 1/3

$$(\partial_x^2 + \partial_y^2 + k^2)u = f \Longrightarrow (-\xi_1^2 - \xi_2^2 + k^2)\hat{u} = \hat{f}$$

$$\hat{u} = \frac{\hat{f}}{-\xi_1^2 - \xi_2^2 + k^2} = \frac{-\hat{f}}{\left(\xi_1 - \sqrt{k^2 - \xi_2^2}\right)\left(\xi_1 + \sqrt{k^2 - \xi_2^2}\right)}$$

$$\hat{u} \text{ ``scattered''} := \frac{-\hat{f}}{2\sqrt{k^2 - \xi_2^2}} \left(\frac{1}{\xi_1 - \sqrt{k^2 - \xi_2^2}} - \frac{1}{\xi_1 + \sqrt{k^2 - \xi_2^2}} \right)$$

where $\sqrt{\dots}$ chosen as a certain branch in $\mathbb{C}$!

New estimates for direct scattering theory

Idea of the proof in 2D 2/3

Result If $\hat{f} \equiv 0$ on $\left|k^2 - \xi_2^2\right| < \delta^2$ then

$$u = f *_{x_1} \mathscr{F}_2^{-1} \frac{ie^{-i\sqrt{k^2-\xi_2^2}|x_1|}}{2\sqrt{k^2-\xi_2^2}},$$

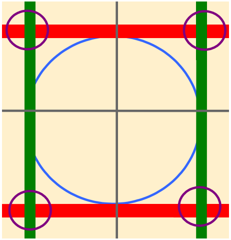
$$\sup_{x_1} \|u\|_{L^2(x_2)} \leq \frac{1}{2\delta} \int_{-\infty}^{\infty} \|f\|_{L^2(x_2)} \, dx_1.$$

Lemma Cut-off's do not cause problems.

New estimates for direct scattering theory

Idea of the proof in 2D 3/3

Lemma A suitable partition of unity exists.



Picture courtesy of J. Sylvester

Corollary If $\operatorname{supp} f \subset \Omega_s$, and $d(\Omega_s) < \infty$ then

$$\|u\|_{L^2(\Omega_w)} \leq \frac{C}{\delta}\sqrt{d(\Omega_w)d(\Omega_s)}\,\|f\|_{L^2(\Omega_s)}$$

for any bounded Ω_w.

For which PDEs will this work?

Inverse backscattering

by
Rakesh (University of Delaware),
Gunther Uhlmann (University of Washington &
HKUST)

Inverse backscattering

Point source backscattering by Rakesh–Uhlmann

Wave generated at $a \in \mathbb{R}^3$, $t = 0$ for potential $q_j \in C_0^\infty(B(0,1))$:

$$\frac{\partial^2}{\partial t^2} U_j^a(x,t) = \left(\Delta_x + q_j(x)\right) U_j^a(x,t) + \delta(x-a,t),$$
$$U_j^a(x,t) = 0, \qquad t < 0$$

Theorem (Rakesh–Uhlmann 2015)

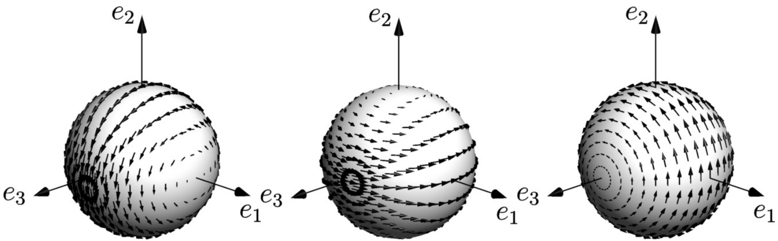
Assume $U_1^a(a,t) = U_2^a(a,t)$ for all $a \in S(0,1)$ and $0 < t < 2$. Then $q_1 = q_2$ under the a-priori assumption of angularly controlled $q_1 - q_2$

$$\|\Omega_{ij}(q_1 - q_2)\|_{L^2(S(0,\rho))} \le C \|q_1 - q_2\|_{L^2(S(0,\rho))}$$

for all $0 < \rho < 1$ and angular derivatives Ω_{ij}.

Angular derivatives Ω_{ij}?

Three vector fields that span the tangent space $TS(0, 1)$:



Why three? -Hairy ball theorem.

Proof idea 1/3

- Formula for boundary measurements

$$U_1^a(a,2\tau) - U_2^a(a,2\tau) = \frac{M(q_1-q_2)(a,\tau)}{8\pi} + \int_{|x-a|\le\tau} (q_1-q_2)(x)k(x,\tau,a)dx$$

where

$$Mf(a,\tau) = \frac{1}{4\pi\tau^2}\int_{S(x,\tau)} f(y)dS_y$$

and k, $\partial_\tau k$ smooth.

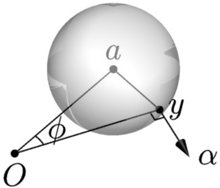
Remark

This is the only part where knowledge of PDE's are required!

Proof idea 2/3

- Formula relating f to Mf

$$f((1-\tau)a) = \frac{2\partial_\tau(\tau Mf(a,\tau))}{1-\tau} + \frac{1}{2\pi(1-\tau)} \int_{S(a,\tau)} \frac{\alpha \cdot \nabla f(y)}{\sin\phi} dS_y$$



$$\left|f((1-\tau)a)\right|^2 \leq \frac{\left|\partial_\tau(\tau Mf(a,\tau))\right|^2 + \sum_{i<j} \int_{S(a,\tau)} \frac{|\Omega_{ij} f(y)|^2}{\sqrt{|y|-(1-\tau)}} dS_y}{(1-\tau)^3}$$

Proof idea 3/3

- Gronwall's inequality

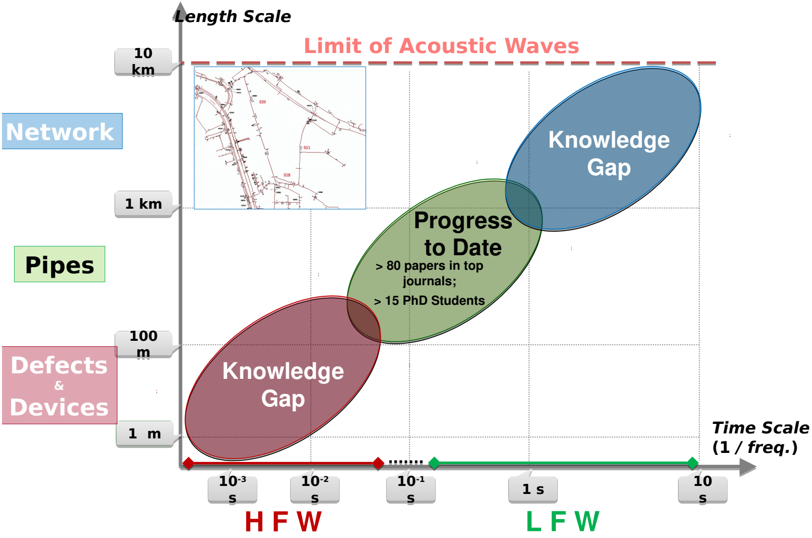
$$f'(s) \leq C \int_s^1 f(r)dr \quad \Longrightarrow \quad f \equiv 0$$

Inverse problem of water pipe systems

with

Mohamed Ghidaoui (HKUST)

Knowledge Gaps



Leaks? Blockages?

Simplest model: Question of 1D wave/transport equation

- Pipe without leaks

$$\frac{A(x)}{c^2}\frac{\partial^2 H}{\partial t^2} - \frac{\partial}{\partial x}\left(A(x)\frac{\partial H}{\partial x}\right) = 0$$

  Solution known since 70's

- Pipe with leak at $x = x_L$ of flow Q_L

$$\begin{cases} \frac{A(x)g}{c^2}\frac{\partial H}{\partial t} + \frac{\partial (A(x)V)}{\partial x} = Q_L \delta(x - x_L) \\ \frac{\partial V}{\partial t} + g\frac{\partial H}{\partial x} = -\frac{V}{A(x)} Q_L \delta(x - x_L) \end{cases}$$

  Q_L depends on $H(x_L)$!!

- Network?

LaTeX-quality mathematics on the web

and the second one by 6. Combined they give us 7.

Theorem 7 (Causal Paley-Wiener theorem for $\mathscr{S}'($
Now supp $u \subset [0, \infty[$ *if and only if there is a unique hol*
constants $M, N \in \mathbb{N}$ *such that*

$$|\langle U(z), \psi \rangle| \leq C \|\psi\|_{M,N} \max(1, |\Im z|^{-M})(1 + |z|)^N$$

for all $\psi \in \mathscr{S}(\mathbb{R}^n)$ *and*

$$\lim_{\sigma \to 0-} U(\cdot + i\sigma) = \mathscr{F}_t u,$$

```
<div class="theorem" id="nDPWthm"
title="Causal Paley-Wiener theorem for $\S'(\R\times\R^n)$"
data-counters="1">
<p>
  Let $u \in \S'(\R\times\R^n)$. Now $\supp u \subset {[{0,\infty}[}$ if
  and only if there is a unique holomorphic $U : \C_- \to \S'(\R^n)$ and
  constants $M,N\in\N$ such that

  \begin{equation}
    \label{Uestimate} \abs{\langle U(z), \psi\rangle} \leq C
    \norm{\psi}_{M,N} \max(1, \abs{\Im z}^{-M}) (1 + \abs{z})^N
  \end{equation}

  for all $\psi\in\S(\R^n)$ and

  \begin{equation}
    \label{Uurelation} \lim_{\sigma\to0-} U(\cdot+i\sigma) = \F_t u,
  \end{equation}
```

Thank you for your attention!