A new viewpoint to scattering theory à la Hörmander

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Collaboration with

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$\mathcal{C}_q \mapsto q$ uniqueness and stability for non-smooth q in 2D

Join work with Oleg Imanuvilov and Masahiro Yamamoto.

$$\mathcal{C}_q = \{(u_{|\Omega}, \partial_\nu u_{|\Omega}) \mid u \in W_2^1(\Omega), (\Delta + q)u = 0\}$$

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and $p > 2$. The following was proven by Bukhgeim (2008) when $q \in W_p^1(\Omega)$.

Theorem (Uniqueness in a domain)

Assume that $q_1, q_2 \in L^p(\Omega)$ with $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$. Then $q_1 = q_2$.

Theorem (Logarithmic stability in a domain)

Let $\varepsilon > 0$ and $M < \infty$. Then there exists constants $C, d_0, \theta > 0$ such that

$$\|q_1 - q_2\|_{L^2(\Omega)} \leq C \left(\ln \frac{1}{d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})} \right)^{-\theta}$$

if $q_1, q_2 \in W_p^{\varepsilon}(\Omega)$ with norms at most M and $d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) \leq d_0$.

$\mathcal{C}_q \mapsto q$ uniqueness and stability for non-smooth q in 2D

Proof idea slide 1/2

Set $\Phi(z) = (z - z_0)^2$ for $z, z_0 \in \mathbb{C}$.

If $u(z) = \mathbf{e}^{\mathbf{i}\tau\boldsymbol{\Phi}(\mathbf{z})} f(z)$ and

$$f = \psi - \frac{1}{4}\overline{\partial}^{-1}\left(e^{-i\tau(\Phi+\overline{\Phi})}\partial^{-1}\left(e^{i\tau(\Phi+\overline{\Phi})} q f\right)\right)$$

with $\Delta(e^{i\tau\Phi}\psi) = 0$, then $(\Delta + q)u = 0$.

Bukhgeim

$$\psi(z) = 1 \quad \forall z$$

Imanuvilov & Yamamoto **According to Uhlmann, these were already defined in http://arxiv.org/abs/1010.5791 by Imanuvilov, Yamamoto & Uhlmann !**

$$\psi(z) = e^{-i\tau(\Phi+\overline{\Phi})} + \frac{\partial^{-1} q(z_0)}{4}\overline{\partial}^{-1}\left(e^{-i\tau(\Phi+\overline{\Phi})}\right)(z)$$

$\mathcal{C}_q \mapsto q$ uniqueness and stability for non-smooth q in 2D

Proof idea slide 2/2

Bukhgeim solutions

$$u_1 = e^{i\tau\Phi} - \frac{1}{4} e^{i\tau\Phi} \overline{\partial}^{-1} \big(e^{-i\tau(\Phi+\overline{\Phi})} \partial^{-1} (e^{i\tau(\Phi+\overline{\Phi})} q_1) \big) + e^{i\tau\Phi} r_1$$

$$u_2 = e^{i\tau\overline{\Phi}} - \frac{1}{4} e^{i\tau\overline{\Phi}} \partial^{-1} \big(e^{-i\tau(\Phi+\overline{\Phi})} \overline{\partial}^{-1} (e^{-i\tau(\Phi+\overline{\Phi})} q_2) \big) + e^{i\tau\overline{\Phi}} r_2$$

Imanuvilov & Yamamoto solutions

$$u_1 = e^{-i\tau\overline{\Phi}} - \frac{1}{4} e^{i\tau\Phi} \overline{\partial}^{-1} \big(e^{-i\tau(\Phi+\overline{\Phi})} (\partial^{-1} q_1 - \partial^{-1} q_1(z_0)) \big) + e^{i\tau\Phi} r_1$$

$$u_2 = e^{-i\tau\Phi} - \frac{1}{4} e^{i\tau\overline{\Phi}} \partial^{-1} \big(e^{-i\tau(\Phi+\overline{\Phi})} (\overline{\partial}^{-1} q_2 - \overline{\partial}^{-1} q_2(z_0)) \big) + e^{i\tau\overline{\Phi}} r_2$$

The second set of solutions give a fast decay for cross-terms in

$0 = \int_\Omega (q_1 - q_2) u_1 u_2$ without the need to integrate by parts.

Fixed energy inverse scattering for non-smooth q in $\mathbb{R}^2$

Joint work with Yang Yang.
Reduce smoothness assumptions of potential in Guillarmou, Salo and Tzou (2010).

Theorem (Uniqueness in $\mathbb{R}^2$ given Alessandrini identity)

Assume that $q_1, q_2 \in L^{(2,1)}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$. Then for any $\varphi \in C_0^\infty(\mathbb{R}^2)$ and $\varepsilon > 0$ we have

$$|\langle q_1 - q_2, \varphi \rangle| \leq \limsup_{\tau \to \infty} \left| \left\langle \int_{\mathbb{R}^2} (q_1 - q_2) u_1 u_2 dm, \varphi \right\rangle \right| + \varepsilon \|\varphi\|_{L^1(\mathbb{R}^2)} .$$

if $u_j \in L^1_{loc}$ are the Imanuvilov & Yamamoto - type CGO solutions.

NOTE: The Alessandrini identity for CGO-solutions requires that q_1 and q_2 have super-exponential decay at infinity.

Fixed energy inverse scattering for non-smooth q in $\mathbb{R}^2$

Difficulties arising from unbounded domain 1/2

First difficulty: Do the Imanuvilov & Yamamoto CGOs exist in the whole $\mathbb{R}^2$? -Yes, they do!

Set $u(z) = e^{i\tau\Phi(z)} f(z)$ with

$$f = \psi - \frac{1}{4}\overline{\partial}^{-1}\left(e^{-i\tau(\Phi+\overline{\Phi})}\partial^{-1}(e^{i\tau(\Phi+\overline{\Phi})} q f)\right), \qquad \tau \neq 0!!!!$$

Choice by B. & Yang

$$\psi(z) = e^{-i\tau(\Phi+\overline{\Phi})} + \frac{\phi(z_0)}{4} u_0(z)$$

with $\phi \in C_0^\infty(\mathbb{R}^2), \|\phi - \partial^{-1} q\| < \epsilon$ and $\overline{\partial} u_0 = e^{-i\tau(\Phi+\overline{\Phi})}$.
Why use u_0 and ϕ? Compare Imanuvilov & Yamamoto:

$$\psi(z) = e^{-i\tau(\Phi+\overline{\Phi})} + \frac{\partial^{-1} q(z_0)}{4}\overline{\partial}^{-1}\left(e^{-i\tau(\Phi+\overline{\Phi})}\right)(z)$$

Fixed energy inverse scattering for non-smooth q in $\mathbb{R}^2$

Difficulties arising from unbounded domain 2/2

Second difficulty: Stationary phase arguments not so trivial. Solved by proving convergence in $\mathscr{D}'(\mathbb{R}^2)$:

$$\lim_{\tau\to\infty}\int_{\mathbb{R}^2}\frac{2\tau}{\pi}Q(z)u_0(z)dm(z) = -\overline{\partial}^{-1}Q(z_0) \quad \text{in } \mathscr{D}'(\mathbb{R}^2).$$

Key to solution: If $\Psi \in C_0^\infty(\mathbb{R}^2)$ then the map

$$\varphi \mapsto \int_{\mathbb{R}^2}\frac{2\tau}{\pi}e^{\pm i\tau(\Phi+\overline{\Phi})}\Psi(z-z_0)\varphi(z)dm(z), \quad \Phi(z) = (z-z_0)^2,$$

extends to a map $E_\Psi : \mathscr{D}'(\mathbb{R}^2) \to \mathscr{D}'(\mathbb{R}^2)$ and moreover

$$\lim_{\tau\to\infty} E_\Psi f = \Psi(0)f \quad \text{in } \mathscr{D}'(\mathbb{R}^2).$$

If $f \in L^2(\mathbb{R}^2)$ then the same is true for $L^2(\mathbb{R}^2)$ instead of $\mathscr{D}'(\mathbb{R}^2)$.

Interesting question: Since we use convergence in $\mathscr{D}'(\mathbb{R}^2)$ at some point in the proof, is it possible to prove stability?

New estimates for direct scattering theory
Background for interest

Join work with John Sylvester.

- So far my research always took $\int (q_1 - q_2) u_1 u_2 = 0$ as a starting point.
- This follows easily in a bounded domain from the definition of equivalent Cauchy data.
- What about for an unbounded domain? Well-known for $\Delta + k^2$, but I wanted to understand more. Leads to a quest for understanding scattering theory more deeply.
- Visited John Sylvester at the UW in spring 2014.
- His 2013 Delaware presentation: new estimates for $(\Delta + k^2)^{-1}$ in 1D and 2D.
- That is the first step in *scattering theory à la Hörmander*.
- Bonus: a new CGO-estimate.

Old well-known estimates

Let $(\Delta + k^2)u = f$. then

- Agmon (1975), $\delta > \frac{1}{2}$

$$\left\|(1+|x|^2)^{-\delta/2}u\right\|_{L^2(\mathbb{R}^n)} \leq \frac{C}{k}\left\|(1+|x|^2)^{\delta/2}f\right\|_{L^2(\mathbb{R}^n)}$$

- Agmon-Hörmander (1976) $A_j = \{2^{j-1} < |x| < 2^{j+1}\}$, $A_0 = \{|x| < 2\}$.

$$\sup_{j\geq 0} \sqrt{2^j}^{-1} \|u\|_{L^2(A_j)} \leq \frac{C}{k}\sum_{j=0}^{\infty} \sqrt{2^j}\,\|f\|_{L^2(A_j)}$$

- Kenig-Ruiz-Sogge (1987) $\frac{1}{q_1} + \frac{1}{q_2} = 1$, $\frac{2}{n+1} \leq \frac{1}{q_1} - \frac{1}{q_2} \leq \frac{2}{n}$

$$\|u\|_{L^{q_2}(\mathbb{R}^n)} \leq Ck^{n(\frac{1}{q_1}-\frac{1}{q_2})-2}\,\|f\|_{L^{q_1}(\mathbb{R}^n)}$$

All of the above not satisfactory from a physical point of view: dilation, rotation, translation, behaviour w.r.t wavelength. . .

New estimates for direct scattering theory

Theorem (J. Sylvester 2013 or earlier)

If $\operatorname{supp} f \subset D_1$ *is bounded then* $(\Delta + k^2)u = f$ *has a scattering solution* u*. It satisfies*

$$\|u\|_{L^2(D_2)} \leq C \frac{\sqrt{\operatorname{diam}(D_2)}\sqrt{\operatorname{diam}(D_1)}}{k} \|f\|_{L^2(D_1)}$$

for any bounded D_2*.*

Corollary

Generalized Agmon-Hörmander estimates:
$f = \sum_j f_j$, $\operatorname{supp} f_j \subset A_j$, *(not necessarily annuli!)*
$u = \sum_j u_j$, $(\Delta + k^2)u_j = f_j$,

$$\sqrt{\operatorname{diam}(D_2)}^{-1} \|u\|_{L^2(D_2)} \leq \frac{C}{k} \sum_j \sqrt{\operatorname{diam}(A_j)} \|f\|_{L^2(A_j)}$$

New estimates for direct scattering theory

Idea of the proof in 1D

$$(D_x^2 + k^2)u = f \Longrightarrow (-\xi^2 + k^2)\hat{u} = \hat{f}$$

$$\hat{u} = -\frac{\hat{f}}{\xi^2 - k^2} = -\frac{\hat{f}}{2k}\left(\frac{1}{\xi - k} - \frac{1}{\xi + k}\right)$$

$$\hat{u} \text{ ``outgoing''} := -\frac{\hat{f}}{2k}\left(\frac{1}{\xi - (k - i0)} - \frac{1}{\xi + (k - i0)}\right)$$

$$\mathscr{F}\{\pm\sqrt{2\pi}i\, H(\pm x)\, e^{izx}\}(\xi) = \frac{1}{\xi - z}, \quad \text{if } \operatorname{sign} \operatorname{Im} z = \pm 1.$$

Result

$$u \text{ ``outgoing''} = f * \frac{ie^{-ik|x|}}{2k}, \qquad \|u\|_{L^\infty} \leq \frac{1}{2k}\|f\|_{L^1}$$

New estimates for direct scattering theory

Idea of the proof in 2D 1/3

$$(D_x^2 + D_y^2 + k^2)u = f \Longrightarrow (-\xi_1^2 - \xi_2^2 + k^2)\hat{u} = \hat{f}$$

$$\hat{u} = \frac{\hat{f}}{-\xi_1^2 - \xi_2^2 + k^2} = \frac{-\hat{f}}{\left(\xi_1 - \sqrt{k^2 - \xi_2^2}\right)\left(\xi_1 + \sqrt{k^2 - \xi_2^2}\right)}$$

$$\hat{u} \text{ ``outgoing''} := \frac{-\hat{f}}{2\sqrt{k^2 - \xi_2^2}} \left(\frac{1}{\xi_1 - \sqrt{k^2 - \xi_2^2}} - \frac{1}{\xi_1 + \sqrt{k^2 - \xi_2^2}} \right)$$

where $\sqrt{k^2 - \xi_2^2}$ chosen to have negative imaginary part! (or $-i0$)

New estimates for direct scattering theory

Idea of the proof in 2D 2/3

Result If $\hat{f} \equiv 0$ on $\left|k^2 - \xi_2^2\right| < \delta^2$ then

$$\mathscr{F}_2 u = \mathscr{F}_1^{-1} \hat{f} *_{x_1} \frac{i e^{-i\sqrt{k^2-\xi_2^2}|x_1|}}{2\sqrt{k^2-\xi_2^2}},$$

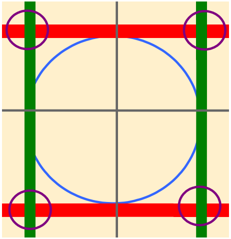
$$\sup_{x_1} \|u\|_{L^2(x_2)} \leq \frac{1}{2\delta} \int_{-\infty}^{\infty} \|f\|_{L^2(x_2)}\, dx_1.$$

Lemma If $\nu \in \mathbb{S}^1$ and $m_\psi f = \mathscr{F}^{-1}\{\psi(\xi - \xi \cdot \nu\nu)\hat{f}(\xi)\}$ for some $\psi \in C_0^\infty(\nu^\perp)$ then

$$\int_{-\infty}^{\infty} \|m_\psi f\|_{L^2(x_2)}\, dx_1 \leq C \int_{-\infty}^{\infty} \|f\|_{L^2(x_2)}\, dx_1.$$

New estimates for direct scattering theory

Idea of the proof in 2D 3/3



Picture courtesy of J. Sylvester

Corollary If $\operatorname{supp} f \subset \Omega_s$, and $d(\Omega_s) < \infty$ then

$$\|u\|_{L^2(\Omega_w)} \leq \frac{C}{\delta}\sqrt{d(\Omega_w)d(\Omega_s)}\,\|f\|_{L^2(\Omega_s)}$$

for any bounded Ω_w.

For which PDEs will this work?

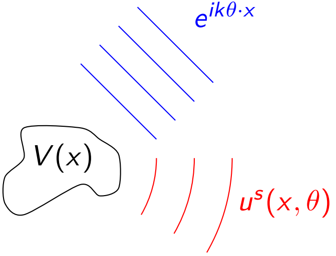
Simple scattering theory for $\Delta + k^2$

Review: time-harmonic plane-wave scattering

Let C_0 be the background wave speed. Model the scatterer with a function V, where

$$1 + V = \frac{C(x)^2}{C_0^2}$$

is the relative speed of wave propagation at the fixed frequency.



The total wave u_θ satisfies

$$\left(\Delta + k^2(1+V)\right) u_\theta = 0,$$

where

$$u_\theta = \underbrace{e^{ik\theta\cdot x}}_{\text{free wave}} + \underbrace{u^s_\theta(x)}_{\text{scattered wave}}$$

New estimates for direct scattering theory

The Hörmander viewpoint

Goal We will show that Sylvester's method and estimate extends to a large class of PDEs.

Solving $P_0(D)u = f$ for all solutions u is the first step to understanding the scattering theory by Agmon and Hörmander. Their theory works for a large class of differential operators

$$P(D) = P_0(D) + V(x, D), \quad D = -i\nabla$$

where the polynomial P_0 is real-valued, non-singular and simply characteristic. The potential V can be any "short-range perturbation", e.g. a differential operator.

New estimates for direct scattering theory

Steps for Agmon-Hörmander scattering

1. study $(P_0(D) - \lambda)^{-1}$, P_0 constant coefficient real polynomial
2. classify all solutions to $(P_0(D) - \lambda)u = f$
3. study spectrum of $P_0(D) - \lambda + V(x, D)$
4. use Fredholm alternative to infer about $(P_0(D) - \lambda + V(x, D))^{-1}$
5. distorted Fourier transform
6. classify all solutions to $(P_0(D) - \lambda + V(x, D))u = 0$

NOTE: we will absorb λ into P_0 in the following slides since it can be kept a constant in steps 1. and 2.

New estimates for direct scattering theory

Idea of general PDE proof in 1D

$$P_0(D)u = f \Longrightarrow P_0(\xi)\hat{u} = \hat{f}$$

$$\hat{u} = \frac{\hat{f}}{P_0(\xi)} = \hat{f} \sum_j \frac{c_j}{\xi - \xi_j}$$ P_0 non-singular, so simple roots!

$$\hat{u} \text{ “regularized” } := \hat{f} \sum_j c_j R_j \frac{1}{\xi - \xi_j}$$ R_j one of *p.v.*, $+i0$ or $-i0$

$$\left\| \mathscr{F}^{-1} \left\{ R_j \frac{1}{\xi - \xi_j} \right\} \right\|_{L^\infty} \leq C$$ no matter which R_j

Result If $\hat{u}$ regularized like above then

$$\|u\|_{L^\infty} \leq C \|f\|_{L^1}$$

New estimates for direct scattering theory

Idea of general PDE proof in 2D 1/3

$$P_0(D)u = f \Longrightarrow P_0(\xi)\hat{u} = P_0(\tau\nu + \xi')\hat{u} = \hat{f}, \quad \nu \in \mathbb{S}^1, \xi' \perp \nu$$

$$\hat{u} = \frac{\hat{f}}{P_0(\xi)} = \hat{f} \sum_j \frac{c_j(\xi')}{\tau - \tau_j(\xi')}$$

$\hat{f} \equiv 0$ on the lines tangent to $P_0^{-1}(0)$!
Note: nontrivial! some τ_j may be real/complex depending on ξ'.

$$\hat{u} \text{ “regularized” } := \hat{f} \sum_j c_j(\xi') R_{j,\xi'} \frac{1}{\tau - \tau_j(\xi')},$$

Note: will the choice of $R_{j,\xi'}$ depend on ξ'? Can it? Anyway

$$\sup_{\xi' \perp \nu} \left\| \mathscr{F}_\nu^{-1} \left\{ R_{j,\xi'} \frac{1}{\tau - \tau_j(\xi')} \right\} \right\|_{L^\infty(\nu\mathbb{R})} \leq C \quad \text{no matter which } R_{j,\xi'}$$

New estimates for direct scattering theory

Idea of general PDE proof in 2D 2/3

Result If $\hat{f} \equiv 0$ whenever $\tau \mapsto \tau\nu + \xi'$ is tangent to $P_0^{-1}(0)$ then

$$\sup_{\nu\mathbb{R}} \|u\|_{L^2(\nu^\perp)} \leq C \int_{-\infty}^{\infty} \|f\|_{L^2(\nu^\perp)} \, ds_{\nu\mathbb{R}}$$

Lemma If $\nu \in \mathbb{S}^1$ and $m_\psi f = \mathscr{F}^{-1}\{\psi(\xi - \xi \cdot \nu\nu)\hat{f}(\xi)\}$ for some $\psi \in C_0^\infty(\nu^\perp)$ then

$$\int_{-\infty}^{\infty} \|m_\psi f\|_{L^2(\nu^\perp)} \, d\nu\mathbb{R} \leq C \int_{-\infty}^{\infty} \|f\|_{L^2(\nu^\perp)} \, d\nu\mathbb{R}.$$

Corollary If the partition of unity succeeds and $\operatorname{supp} f \subset \Omega_s$ with $d(\Omega_s) < \infty$ then there is u s.t. $P_0(D)u = f$. Moreover

$$\|u\|_{L^2(\Omega_w)} \leq C\sqrt{d(\Omega_w)d(\Omega_s)} \, \|f\|_{L^2(\Omega_s)}$$

for any bounded Ω_w.

New estimates for direct scattering theory

Idea of general PDE proof in 2D 3/3

When will the partition of unity succeed?

Difficulties

- P_0 of arbitrarily high degree
- geometric tangent VS algebraic tangent ($\mathbb{R}^n$ VS $\mathbb{C}^n$)

Under what assumptions can we do it currently

- $P_0 : \mathbb{C}^2 \to \mathbb{C}$ (coefficients may be complex)
- P_0 uniformly non-singular ($P_0(\xi) = 0 \Rightarrow |\nabla P_0(\xi)| \geq g_0 > 0$)
- When $|\xi| \to \infty$ along $P_0(\xi) = 0$ we have

$$\frac{\xi}{|\xi|} \cdot \frac{\nabla P_0(\xi)}{|\nabla P_0(\xi)|} \longrightarrow 0 \qquad \text{(no twirling to infinity)}$$

- For all ξ there is $\nu \in \mathbb{S}^1$ such that the polynomial

$$\tau \mapsto P_0(\tau\nu + \xi)$$

  has simple roots and is of same degree as P_0

Inverse problem for Agmon-Hörmander scattering

Setting: If P_0 is simply characteristic, V is its short-range perturbation and $\lambda \in \mathbb{R}$ avoids a discrete set, then the solutions to

$$(P_0(D) - \lambda - V(x, D))u = 0$$

can be split into free wave + scattered wave. Moreover the **scattering matrix** Σ_λ is well defined.

Inverse problem: does Σ_λ determine P_0 and V?

- Is there any hope? What kind of counterexamples?
- Does it at least determine the degree of P_0?
- Can the problem be solved if we know Σ_λ for many λ?

New estimates for direct scattering theory

BONUS: new CGO-estimate

Theorem

There is $C>0$ such that if $\rho\in\mathbb{C}^n$, $\rho\cdot\rho=0$ then

$$\left\|(\Delta-2\rho\cdot\nabla)^{-1}f\right\|_{L^\infty(\Re\rho,\widehat{L^{p_2}}(\Re\rho^\perp))} \leq C\,|\rho|^{(n-1)(\frac{1}{p_2}-\frac{1}{p_1})-1}\,\|f\|_{L^1(\Re\rho,\widehat{L^{p_1}}(\Re\rho^\perp))}$$

when $\frac{1}{p_2}-\frac{1}{p_1}<\frac{1}{n-1}$ and $p_2\leq p_1$. No need for $\frac{1}{p_1}+\frac{1}{p_2}=1$!!!

Corollary

If $\operatorname{supp} f\subset\Omega_s$, $\frac{1}{q_1}-\frac{1}{q_2}<\frac{1}{n-1}$ and $q_1\leq 2\leq q_2$ then

$$\left\|(\Delta-2\rho\cdot\nabla)^{-1}f\right\|_{L^{q_2}(\Omega_w)} \leq C d(\Omega_w)^{1/q_2} d(\Omega_s)^{1/q_1}\,|\rho|^{(n-1)(\frac{1}{q_1}-\frac{1}{q_2})-1}\,\|f\|_{L^{q_1}(\Omega_s)}\,.$$

New estimates for direct scattering theory

New CGO-estimate proof

- $\rho = R + iI$, $R, I \in \mathbb{R}^n$, $|R| = |I|$, $I \perp R$
- $R = s\nu$, $s > 0$, $\nu \in \mathbb{S}^{n-1}$
- $\xi = \tau\nu + \xi'$

Then we can split

$$\frac{1}{-|\xi|^2 - 2i\rho\cdot\xi} = \frac{-1}{\big(\tau + i(s + |\xi' - I|)\big)\big(\tau + i(s - |\xi' - I|)\big)}$$

The operator

$$Ag := \mathscr{F}^{-1}\left\{p.v.\frac{\hat{g}}{\tau + ib(\xi')}\right\}$$

maps

$$\|Ag\|_{L^\infty(\Re\rho, \widehat{L^{p_2}}(\Re\rho^\perp))} \le C(\inf b)^{\alpha(n,p_1,p_2)}\|g\|_{L^\infty(\Re\rho, \widehat{L^{p_1}}(\Re\rho^\perp))},$$

$$\|Ag\|_{L^\infty(\Re\rho, \widehat{L^p}(\Re\rho^\perp))} \le C\|g\|_{L^1(\Re\rho, \widehat{L^p}(\Re\rho^\perp))}.$$

Thank you for your attention!