Non-scattering energies and interior transmission eigenvalues
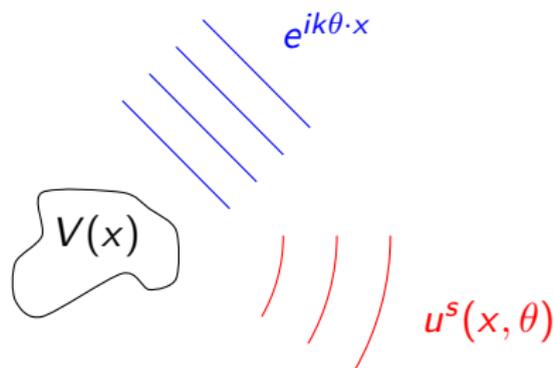
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Single Frequency Scattering Theory

The total wave u satisfies

$$(\Delta + k^2(1 + V))u = 0,$$

where V models a perturbation to the background wave speed and

$$u = e^{ik\theta \cdot x} + u^s(x, \theta)$$

incident wave   scattered wave

1) Single incident plane wave

$$u^s(x, \theta) = \frac{e^{ik|x|}}{|x|} A(\hat{x}, \theta) + O(\frac{1}{|x|^2}),$$

where $A(\hat{x}, \theta)$ is the scattering amplitude.

2) General incident wave $u_g^i =$ superposition of plane waves $e^{ik\theta \cdot x}$

$$\underbrace{\int_{\mathbb{S}^{n-1}} u(x, \theta) g(\theta) d\theta}_{u_g(x)} = \underbrace{\int_{\mathbb{S}^{n-1}} e^{ik\theta \cdot x} g(\theta) d\theta}_{u_g^i(x)} + \underbrace{\int_{\mathbb{S}^{n-1}} u^s(x, \theta) g(\theta) d\theta}_{u_g^s(x)}$$

Amplitude of general scattered wave = superposition of single plane scattering amplitudes

$$A_g(\theta') = \int_{\mathbb{S}^{n-1}} A(\theta', \theta) g(\theta) d\sigma(\theta)$$

Vanishing Scattering Amplitude?

Question: Can there be $g \in L^2(\mathbb{S}^{n-1})$, $g \neq 0$ such that $A_g \equiv 0$?

Consequence:

$$\text{Rellich's theorem} \implies u_g^s \equiv 0 \quad \mathbb{R}^n \setminus \operatorname{supp} V.$$

Recall that

$$u_g = \underbrace{u_g^i}_{w} + \underbrace{u_g^s}_{v} \longleftarrow \text{compact support}$$

Now v and w satisfy

$$
\begin{array}{llll}
(\Delta + k^2(1+V))w = 0 & \text{in } \Omega & w = v & \text{on } \partial\Omega \\
(\Delta + k^2)v = 0 & \text{in } \Omega & \frac{\partial}{\partial \nu}w = \frac{\partial}{\partial \nu}v & \text{on } \partial\Omega
\end{array}
$$

This is called the Interior Transmission Problem (ITP). If there is such $w, v \neq 0$ then k is an interior transmission eigenvalue (ITE).

Interior Transmission Problem

Why interesting:

- generalized eigenvalue problem (analytic Fredholm theory)
- correspond to resonant frequencies of impenetrable scatterers
- ITE's show up in the far field data
- can V be determined from ITP spectrum?

Some history:

- 86', 88' **Kirsch, Colton–Monk**: ITP posed
- 89', 91' **Colton–Kirsch–Päivärinta, Rynne–Sleeman**: discreteness of ITE
- 91'–08' NOTHING. . .
- 07', 09' **Cakoni–Colton–Monk, Cakoni–Colton–Haddar**: qualitative information about V from ITE's
- 08' **Päivärinta–Sylvester**: existence for general scatterers
- 10' **Cakoni–Gintides–Haddar**: infinitely many ITE's
- 10' **Cakoni–Colton–Haddar**: ITE's can be deduced from far-field data
- 10'+: EXPLOSION OF INTEREST

Interior Transmission Problem

What kind of things are known nowadays:

- ITE's generally form an infinite discrete set
- generalized eigenfunctions form a complete set
- Weil laws for the ITE counting function
- numerical algorithms
- ITP solved in many situations: rough potential, rough domain, higher order PDE's, Maxwell, elasticity, unbounded domain...

Some people publishing in the field: Kirsch, Colton, Monk, Päivärinta, Cakoni, Sylvester, Haddar, Robbiano, Lakshtanov, Vainberg, Bonnet-Ben Dhia, Chesnel, Delbary, Hu, Salo, Vesalainen, Selgas, Leung, Gintides, Pallikarakis, Sun, Xu, Harris, Chen, Meng, ...

Survey (2013): CAKONI, HADDAR: *Transmission eigenvalues in inverse scattering theory*, in Inverse Problems and Applications: Inside Out II, 529–578.

Non Scattering Energies

Definition
$\lambda > 0$ is a non-scattering energy (NSE) if the scattering amplitude is not injective, i.e. there is an incident wave

$$u_g(x) = \int_{\mathbb{S}^{n-1}} e^{ik\theta \cdot x} g(\theta) d\sigma(\theta), \quad k = \sqrt{\lambda},$$

such that the scattered wave u_s has zero far field, so $A_g \equiv 0$.

Remark
Non-scattering energies are transmission eigenvalues.

Theorem
{Transmission eigenvalues} $\neq$ {non-scattering energies}
(B.–Päivärinta–Sylvester 2014, Päivärinta–Salo–Vesalainen submitted 2014)

Results for non scattering energies

Theorem (B.–Päivärinta–Sylvester, 2014)

Let $V = \chi_C \varphi$, $C =]0, \infty[^n$ and φ with bounded support and $\varphi(\bar{0}) \neq 0$. Then V scatters all incident waves at all energies.

Theorem (Päivärinta–Salo–Vesalainen, submitted 2014)

$V = \chi_C \varphi$, $C \subset \mathbb{R}^n$, $n \in \{2, 3\}$ a circular cone with angle $< \pi$ avoiding a discrete set of a-priori forbidden angles. Then no NSE's.

Theorem (Bonnet-Ben Dhia–Chesnel–Nazarov, 2015)

Non-scattering energies when far field known in a given **finite** set of directions. Then NSE form a discrete set. Given energy λ and domain Ω construct V in Ω such that λ is a NSE.

Theorem (Hu–Salo–Vesalainen, submitted 2015)

Single incident wave determines shape and location of polygonal penetrable scatterer in $\mathbb{R}^2$. Same for rectangular scatterers in $\mathbb{R}^3$.

Results used in the proof of our theorem

Proposition

If v is a non-scattering incident wave, then

$$\int Vvw dx = 0$$

for all $w \in L^1_{loc}$, $(\Delta + k^2(1+V))w = 0$.

Proposition

Let H be a homogeneous harmonic polynomial. If $\int_{x \geq 0} e^{\rho \cdot x} H(x)dx = 0$ for all $\rho \in \mathbb{C}^n$, $\rho \cdot \rho = 0$, $\Re\rho < 0$, then $H \equiv 0$.

Proposition

For such ρ with $|\Im\rho| > r_0$ there is $w \in L^p_{loc}$, $2 \leq p < \infty$ such that

$$w(x) = e^{\rho \cdot x}(1 + \psi(x)), \qquad \|\psi\|_{L^p(\Omega)} \leq C_\Omega |\Im\rho|^{-1}.$$

Proof overview 1/2

$$\int_C \varphi vw dx = 0$$

Assume for simplicity that $\varphi \equiv 1$ in its support which is in $]0,1[^n$.
Let w be a CGO solution. By the proposition on the previous slide

$$\int_{]0,1[^n} e^{\rho \cdot x}(1 + \psi(x))v(x)dx = 0.$$

Since $\int_{|x|>1} e^{x \cdot \rho}(1 + \psi)v \, dx$ decays exponentially as $|\rho| \to \infty$, we get

$$\left| \int_{x \geq 0} e^{x \cdot \rho}(1 + \psi(x))v(x)dx \right| \leq \exp\left(-c|\rho|\right).$$

Lemma
Let $(\Delta + \lambda)v = 0$ in $\mathbb{R}^n$. Then the principal term P_N in the Taylor-expansion of v is a homogeneous harmonic polynomial or order N.

Proof overview 2/2

Write $v = P_N + v_{N+1}$, where v_{N+1} is of order $N + 1$. Then

$$\left| \int_{x \geq 0} e^{\rho \cdot x} (1 + \psi(x)) v_{N+1}(x) dx \right| \leq C |\rho|^{-N-n-1}$$

and

$$\left| \int_{x \geq 0} e^{\rho \cdot x} \psi(x) P_N(x) dx \right| \leq C |\rho|^{-N-n+n/p} \|\psi\|_p.$$

Using $\|\psi\|_p \leq C |\rho|^{-1}$ we end up with

$$\left| \int_{x \geq 0} e^{\rho \cdot x} P_N(x) dx \right| \leq C |\rho|^{-N-n-1+n/p}$$

as $|\rho| \to \infty$. LHS $\approx C |\rho|^{-N-n}$. Hence

$$\int_{x \geq 0} e^{\frac{\rho}{|\rho|} \cdot x} P_N(x) dx \equiv 0.$$

so $P_N \equiv 0$. $\qquad\qquad\square$

Problem 1: CGO with fast error decay

Earlier estimates for the Faddeev Green's function 1/2

Definition
Let

$$\mathscr{G}_\rho f := \mathscr{F}^{-1}\left(\frac{-\hat{f}}{|\xi|^2 - 2i\rho \cdot \xi}\right), \qquad \rho \cdot \rho = 0$$

Theorem (Well known Agmon estimate)

$$\|\mathscr{G}_\rho f\|_{L^2_{\delta+1}(\mathbb{R}^n)} \leq \frac{C}{|\rho|}\|f\|_{L^2_\delta(\mathbb{R}^n)}, \qquad -1 < \delta < 0$$

It implies in that

$$\|\psi\|_{L^2(\Omega)} \leq C |\rho|^{-1}.$$

But then we needed to have $n/2 - 1 < 0$ for the proof !!

Problem 1: CGO with good error decay

The best sharp result for $L^p(\mathbb{R}^n)$ is

Theorem (Kenig–Ruiz–Sogge)

If
$$\frac{1}{p} + \frac{1}{q} = 1 \quad and \quad \frac{2}{n+1} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{2}{n},$$

then $\mathscr{G}_\rho : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ *with*

$$\|\mathscr{G}_\rho f\|_q \leq C \, |\rho|^{n(\frac{1}{p} - \frac{1}{q}) - 2} \, \|f\|_p \, .$$

This is not enough decay for the proof!

And this result is sharp. Cannot get better in $L^p(\mathbb{R}^n)$!

Problem 1: CGO with good error decay

Why not prove the estimate on the Fourier side?

Theorem (B.–Päivärinta–Sylvester)

The following Agmon-Hörmander style estimate holds:

$$\|\mathscr{F}\mathscr{G}_\rho f\|_{B_{p,\infty}^{-1}} \leq C |\rho|^{-1} \|\mathscr{F}f\|_{B_{p,1}^1}, \qquad 1 \leq p \leq \infty$$

Corollary

There is a CGO solution $e^{\rho \cdot x}(1 + \psi)$ with $\|\psi\|_p \leq C |\rho|^{-1}$ locally for $2 \leq p < \infty$.

Proof.

The error term satisfies $(\Delta + 2\rho \cdot \nabla)\psi = V + V\psi$ so

$$\mathscr{F}\psi = \mathscr{F}\mathscr{G}_\rho V + \mathscr{F}\mathscr{G}_\rho V\psi.$$

Also $V = \prod_{j=1}^n H(x_j)\varphi$ with φ smooth enough. So $\mathscr{F}V\psi \in B_{q,1}^1$ if $\mathscr{F}\psi \in B_{q,\infty}^{-1}$. Finally, $\mathscr{F}^{-1}B_{q,\infty}^{-1} \hookrightarrow L_{loc}^p$ if $p^{-1} + q^{-1} = 1$. $\qquad\square$

Problem 1: CGO with good error decay
Proof of the new estimate

Lemma
Let $\mathscr{M} = \{x \mid g(x) = 0\}$ be a codimension $k \geq 2$ compact manifold and $|\nabla g| \geq 1$ on $\mathscr{M}$. Then

$$\left\| \frac{1}{g} * \chi_\epsilon \right\|_\infty \leq C\epsilon^{-1}$$

for $1 > \epsilon > 0$, $\chi_\epsilon(x) = \epsilon^{-n}\chi(x/\epsilon)$.

This general result implies a uniform bound for the Fourier-transforms of the cut-offs of the Faddeev kernel:

Corollary

$$\left\| \frac{1}{-|\xi|^2 + 2i\rho \cdot \xi} * \chi_\epsilon \right\|_\infty \leq \frac{C}{|\rho|}\epsilon^{-1}$$

Problem 2: Laplace transform vanishing on submanifold

Proposition

Let $H : \mathbb{C}^n \to \mathbb{C}$ be a homogeneous harmonic polynomial. If

$$\int_{x \geq 0} e^{\rho \cdot x} H(x) dx = 0$$

for all $\rho \in \mathbb{C}^n$, $\rho \cdot \rho = 0$, $\Re \rho < 0$, then $H \equiv 0$.

A direct integration gives

$$\int_{x \geq 0} e^{x \cdot \rho} H(x) dx = P(\tfrac{1}{\rho}) \qquad \left(\frac{1}{\rho}\right)_j := \frac{1}{\rho_j}.$$

Integration by parts (using $\Delta H = 0$) gives

$$\int_{x \geq 0} e^{x \cdot \rho} H(x) dx = \frac{1}{\rho \cdot \rho} Q(\tfrac{1}{\rho})$$

where P, Q are homogeneous polynomials.

Write $z_j = 1/\rho_j$. Then (if $n = 3$)

$$\rho \cdot \rho = \frac{1}{z} \cdot \frac{1}{z} = \frac{z_1^2 z_2^2 + z_2^2 z_3^2 + z_3^2 z_1^2}{z_1^2 z_2^2 z_3^2}.$$

$1/(\rho \cdot \rho)Q(1/\rho) = 0$ on $\rho \cdot \rho = 0$ and Hilbert's Nullstellensatz imply that

$$Q(z) = C(z)(z_1^2 z_2^2 + z_2^2 z_3^2 + z_3^2 z_1^2)^2$$

Problem 2: Laplace transform vanishing on submanifold

Polynomial algebra

Integration by parts $\Longrightarrow$ terms of Q do not have $z_1^2 z_2^2 z_3^2$ as factor.

Lemma
Let Q be a homogeneous polynomial without terms having $z_1^2 z_2^2 z_3^2$ as a factor. Then Q does not have $F^2 = (z_1^2 z_2^2 + z_2^2 z_3^2 + z_3^2 z_1^2)^2$ as a factor unless $Q \equiv 0$.
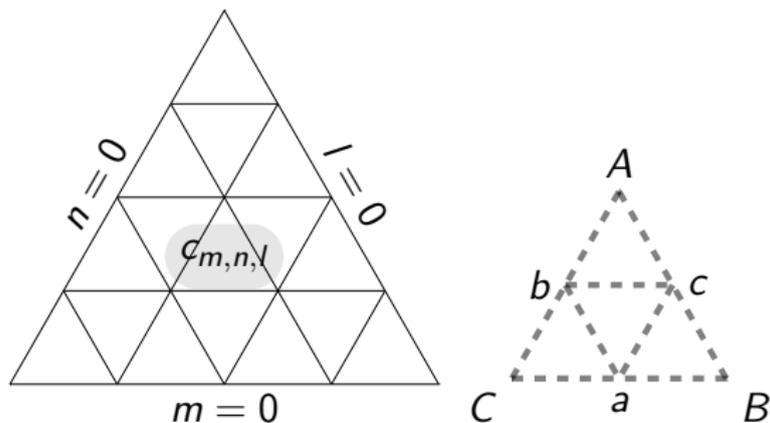
Proof.
Expand $Q = CF^2$, equate sum of terms having $z_1^2 z_2^2 z_3^2$ as zero.
Gives a system of linear equations proving $C \equiv 0$. □

- C homogeneous
- $c_{m,n,l}$ its coefficients. Represented as nodes on the left.
- A single equation is represented by sliding the right triangle onto the left one, both sharing ≥ 3 common nodes.
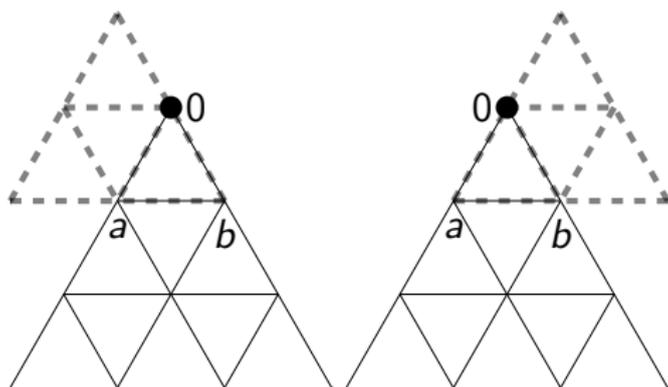

$$A + B + C + 2a + 2b + 2d = 0$$

- One of the equations shows that the top node $c_{m,0,0} = 0$.
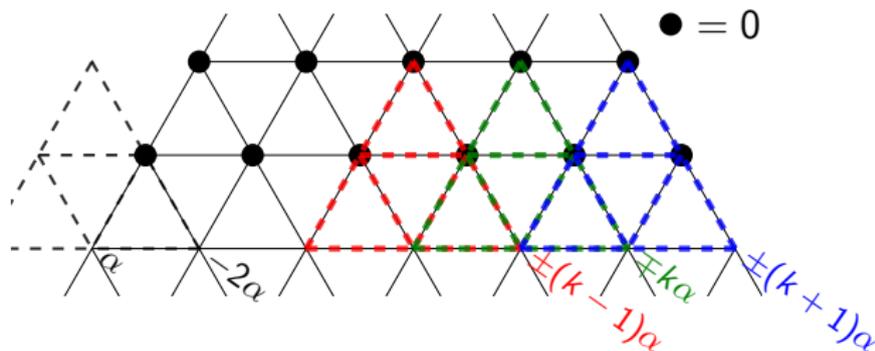- The next line $a = c_{m-1,0,1}$, $b = c_{m-1,1,0}$:

$$\left.\begin{array}{r} 2a + b = 0 \\ a + 2b = 0 \end{array}\right\} \Rightarrow a = b = 0$$

Problem 2: Laplace transform vanishing on submanifold

- Induction: Let the leftmost node be α.
- Then the next one must be -2α.
- ...

- In the end

$$\mp k\alpha \pm 2(k+1)\alpha = 0 \implies \alpha = 0.$$

- By induction $c_{m,n,l} = 0 \quad \forall m, n, l.$

Q.E.D.

Theorem (B.–Päivärinta–Sylvester 2014)

Let $V = \chi_C \varphi$, $C =]0, \infty[^n$ and φ with bounded support and $\varphi(\bar{0}) \neq 0$. Then V scatters all incident waves at all energies.

From high-school algebra to non-trivial inverse scattering

Conjecture

Let $P : \mathbb{C}^n \to \mathbb{C}$ be a homogeneous second degree polynomial. Let H be a homogeneous polynomial satisfying $P(\nabla)H = 0$. If

$$\int_{x \geq 0} e^{\rho \cdot x} H(x) dx = 0$$

for $P(\rho) = 0$, $\Re\rho < 0$ then $H \equiv 0$.

Corollary

Polygonal **penetrable** scatterers always scatter and their shape can be identified from a single measurement.

Proof.

2D okay: HU–SALO–VESALAINEN, *Shape identification in inverse medium scattering problems with a single far-field pattern*. $\qquad\square$

Impenetrable: LIU–PETRINI–RONDI–XIAO, *Stable determination of sound-hard polyhedral scatterers by a minimal number of scattering measurements*.

Non-rectangular corners?

Theorem (Päivärinta–Salo–Vesalainen, 2014)

- $C \subset \mathbb{R}^2$ *strictly convex cone with vertex* $\bar{0}$,
- $\varphi : \mathbb{R}^2 \to \mathbb{R}$ *superexponentially decaying*, $\varphi(\bar{0}) \neq 0$,
- $\varphi \in C^s(\mathbb{R}^2)$ *for some* $s > 0$.

Then $V = \chi_C \varphi$ *has no non-scattering energies.*

Theorem (Päivärinta–Salo–Vesalainen, 2014)

- $C \subset \mathbb{R}^3$ *strictly convex circular cone with vertex* $\bar{0}$,
- *opening angle of* $C \notin E$, *where* E *a-priori given and countable*,
- $\varphi : \mathbb{R}^3 \to \mathbb{R}$ *superexponentially decaying*, $\varphi(\bar{0}) \neq 0$,
- $\varphi \in C^s(\mathbb{R}^3)$ *for some* $s > \frac{1}{4}$.

Then $V = \chi_C \varphi$ *has no non-scattering energies.*

Proof ideas

- Kenig-Ruiz-Sogge type estimate for CGO construction: If

$$\frac{1}{p} + \frac{1}{q} = 1 \quad \text{and} \quad \frac{2}{n+1} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{2}{n},$$

then $\mathscr{G}_\rho : H^{s,p}(\mathbb{R}^n) \to H^{s,q}(\mathbb{R}^n)$ with

$$\|\mathscr{G}_\rho f\|_{s,q} \leq C \, |\rho|^{n(\frac{1}{p} - \frac{1}{q}) - 2} \, \|f\|_{s,p}.$$

- polar coordinates + spherical harmonics decomposition of H to show that $H \equiv 0$ if Laplace transform vanishes:

$$\int_{C \cap \mathbb{S}^{n-1}} \int_0^\infty e^{\rho \cdot \theta r} r^{N+n-1} H(\theta) \, dr \, d\theta = 0$$

□

Work almost submitted

Theorem (B.–Pohjola–Vesalainen)

In the hyperbolic space $\mathbb{H}^n$ hyperbolic rectangular ($n \in \mathbb{N}$) or spherical ($n \in \{2, 3\}$) penetrable cones always scatter.

Open problem

Characterise all domains/potentials that have non-scattering energies.

Conjecture

Only radially symmetric potentials have non-scattering energies.

Thank you for your attention