Solving the Inverse Problem for the 2D Schrödinger Equation with L^p-potential

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Collaboration with Oleg Imanuvilov, Masahiro Yamamoto and Yang Yang
+many giants on whose shoulders we stand

Inverse problems for partial differential equations

The Calderón problem: given an open set $\Omega \subset \mathbb{R}^n$ and all (voltage, current flux) pairs $(v, f) \in H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ satisfying

$$\begin{aligned} \nabla \cdot \gamma \nabla u &= 0 \quad && \Omega \\ u &= v && \partial\Omega \\ \gamma \partial_\nu u &= f && \partial\Omega \end{aligned}$$

deduce the conductivity γ inside Ω. Asked by Calderón and he solved the linearised problem.

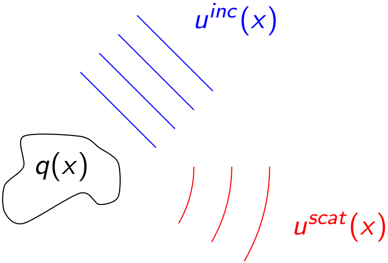
Inverse potential scattering

Another well-studied problem is inverse scattering: given

$$\mathcal{C}_q = \left\{ (u_{|\partial\Omega}, \partial_\nu u_{|\partial\Omega}) \,\middle|\, (\Delta + q)u = 0, u \in H^1(\Omega) \right\},$$

deduce the scattering potential q inside Ω.

With potential reaching infinity: given the scattering matrix $S_q(\lambda)$ for a fixed frequency or wavenumber λ, determine q.



$$(\Delta + q + \lambda^2)(u^{inc} + u^{scat}) = 0$$

$S_q(\lambda)$ relates the behaviours of u^{inc} and u^{scat} at infinity.

Some papers before Bukhgeim

- Calderón 1980 (manuscript from 60's): linearised problem
- Kohn & Vogelius 1984: piecewise analytic γ
- Sylvester & Uhlmann 1987: arbitrary smooth γ and q in 3D
- Alessandrini 1988: logarithmic stability result
- Astala & Päivärinta 2006: arbitrary $\gamma \in L^\infty(\Omega)$ in 2D
- Bukhgeim 2008: $\Delta + q$ with $q \in W^{1,p}(\Omega)$ in 2D
- Novikov & Santacesaria 2010: stability for $q \in C^2(\Omega)$ in 2D
- Guillarmou, Salo & Tzou 2010: uniqueness for $q \in C^{1,\alpha}(\mathbb{R}^2) \cap e^{-c|z|^2} L^\infty(\mathbb{R}^2) \quad \forall c > 0$

Also, among others: Nachman, Liu, Jerison, Kenig, ...

Our main results for low smoothness potentials

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and $p > 2$.

Theorem (Uniqueness in a domain)

Assume that $q_1, q_2 \in L^p(\Omega)$ *with* $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$. *Then* $q_1 = q_2$.

Theorem (Logarithmic stability in a domain)

Let $\varepsilon > 0$ *and* $M < \infty$. *Then there exists constants* $C, d_0, \theta > 0$ *such that*

$$\|q_1 - q_2\|_{L^2(\Omega)} \leq C \left(\ln \frac{1}{d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})} \right)^{-\theta}$$

if $q_1, q_2 \in W_p^\varepsilon(\Omega)$ *with norms at most* M *and* $d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) \leq d_0$.

Theorem (Uniqueness in $\mathbb{R}^2$, yet to be submitted)

Assume that $q_1, q_2 \in L^{(2,1)}(\mathbb{R}^2) \cap e^{-c|z|^2} L^1(\mathbb{R}^2) \quad \forall c > 0$ *with* $S_{q_1}(0) = S_{q_2}(0)$. *Then* $q_1 = q_2$.

We will give the general idea on how to prove these results of 2D inverse problems.

First step in solving potential scattering inverse problems

If q_1 and q_2 give the same measurement results, then

$$\int (q_1 - q_2) u_1 u_2 dm = 0$$

for all admissible u_j satisfying

$$(\Delta + q_j) u_j = 0.$$

Complex Geometric Optics solutions give information about $q_1 - q_2$ if we know $\int (q_1 - q_2) u_1 u_2$:

$$u(x) = e^{\rho \cdot x}(1 + \varepsilon(x)), \quad \rho \in \mathbb{C}^n, \quad \rho \cdot \rho = 0,$$

or

$$u(z) = e^{i\tau\Phi(z)}(1 + \varepsilon(x)), \quad \overline{\partial}\Phi = 0, \quad \tau \gg 1.$$

How to use Complex Geometric Optics solutions 3D+

Idea of Sylvester and Uhlmann (1987):
In n dimensions, $n \geq 3$, for any $\xi \in \mathbb{R}^n$ we may choose $\rho, \rho' \in \mathbb{C}^n$ with $\rho \cdot \rho = \rho' \cdot \rho' = 0$ and

$$\rho = i(\xi + a) + b, \quad \rho' = i(\xi - a) - b,$$

$$\xi, a, b \in \mathbb{R}^n, \quad a \perp \xi \perp b \perp a, \quad |b|^2 = |\xi|^2 + |a|^2$$

Not possible in 2D!

Then if $|a|$ is large enough, CGO solutions $u_1(x) = (1 + \varepsilon_1(x))e^{\rho \cdot x}$ and $u_2(x) = (1 + \varepsilon_2(x))e^{\rho' \cdot x}$ exist, and then

$$\begin{aligned} 0 = \int (q_1 - q_2)(x) u_1(x) u_2(x) dm(x) \\ = \int (q_1 - q_2)(x) e^{2i\xi \cdot x} \left(1 + O(|a|^{-1})\right) dm(x) \end{aligned}$$

Letting $|a| \to \infty$ we have $\mathscr{F}\{q_1 - q_2\}(2\xi) = 0$. $\square$

How to use Complex Geometric Optics solutions 2D

20 years later: Bukhgeim's idea for 2D: stationary phase method!

$$\lim_{\tau\to\infty} \frac{2\tau}{\pi} \int_{\mathbb{C}} e^{i\tau((z_0-z)^2+(\overline{z_0}^2-\overline{z}^2))}(q_1-q_2)(z)dm(z) = (q_1-q_2)(z_0)$$

New types of CGO's:

$$u_1(z) = C\sqrt{\tau}e^{i\tau(z_0-z)^2}(1+\varepsilon_1(z)), \qquad (\Delta+q_1)u_1 = 0$$

$$u_2(z) = C\sqrt{\tau}e^{i\tau(\overline{z_0}-\overline{z})^2}(1+\varepsilon_2(z)), \qquad (\Delta+q_2)u_2 = 0$$

If $\sup_z |\varepsilon_1(z)|, |\varepsilon_2(z)| = O(\tau^{-\frac{1}{2}})$ then after some work

$$\begin{aligned} 0 &= \int (q_1-q_2)(z)u_1(z)u_2(z)dm(z) \\ &= \frac{2\tau}{\pi}\int e^{i\tau((z_0-z)^2+(\overline{z_0}^2-\overline{z}^2))}(q_1-q_2)(z)dm(z) + \text{small error} \end{aligned}$$

Handling the error requires tricks! Integration by parts for ex. $\square$

How to solve for the CGO solutions in 2D?

Complex derivative operators

$$\partial = \frac{1}{2}(\partial_x - i\partial_y) \qquad \overline{\partial} = \frac{1}{2}(\partial_x + i\partial_y) \qquad \Delta = 4\partial\overline{\partial}$$

Write $\Phi(z) = (z_0 - z)^2$. Then (defining g in the middle)

$$-qe^{i\tau\Phi} = \Delta(e^{i\tau\Phi} f) = 4\partial\big(e^{i\tau\Phi}\overline{\partial} f\big) = 4\partial\big(e^{-i\tau\overline{\Phi}} g\big) = 4e^{-i\tau\overline{\Phi}}\partial g$$

hence

$$\begin{cases} \overline{\partial} f = e^{-i\tau(\Phi+\overline{\Phi})} g \\ \partial g = -\frac{1}{4} q e^{i\tau(\Phi+\overline{\Phi})} f \end{cases} \implies (\Delta + q)(e^{i\tau\Phi} f) = 0.$$

Next: use the right inverses

$$\overline{\partial}^{-1} h(z) = \frac{1}{\pi}\int \frac{h(z')}{z - z'} dm(z'), \qquad \partial^{-1} h(z) = \frac{1}{\pi}\int \frac{h(z')}{\overline{z} - \overline{z'}} dm(z').$$

Integral equation for CGO solutions

If $u(z) = C\sqrt{\tau}e^{i\tau\Phi(z)}f(z)$ and

$$f = \psi - \frac{1}{4}\overline{\partial}^{-1}\big(e^{-i\tau(\Phi+\overline{\Phi})}\partial^{-1}(e^{i\tau(\Phi+\overline{\Phi})}qf)\big)$$

with $\Delta(e^{i\tau\Phi}\psi) = 0$, then $(\Delta + q)u = 0$.
Bukhgeim

$$\psi(z) = 1 \quad \forall z$$

Imanuvilov & Yamamoto

$$\psi(z) = e^{-i\tau(\Phi+\overline{\Phi})} + \frac{\partial^{-1}q(z_0)}{4}\overline{\partial}^{-1}\big(e^{-i\tau(\Phi+\overline{\Phi})}\big)(z)$$

Blåsten & Yang

$$\psi(z) = e^{-i\tau(\Phi+\overline{\Phi})} + \frac{\phi(z_0)}{4}u_0(z)$$

with $\phi \in C_0^\infty(\mathbb{R}^2), \|\phi - \partial^{-1}q\| < \epsilon$. Reasons for using u_0 and ϕ?

Explanation 1: choice by Imanuvilov & Yamamoto

$$\frac{u_1}{C\sqrt{\tau}} = e^{-i\tau\overline{\Phi}} + \frac{1}{4}e^{i\tau\Phi}\overline{\partial}^{-1}\left(e^{-i\tau(\Phi+\overline{\Phi})}(\partial^{-1}q_1(z_0) - \partial^{-1}q_1)\right) + e^{i\tau\Phi}r_1$$

$$\frac{u_2}{C\sqrt{\tau}} = e^{-i\tau\Phi} + \frac{1}{4}e^{i\tau\overline{\Phi}}\partial^{-1}\left(e^{-i\tau(\Phi+\overline{\Phi})}(\overline{\partial}^{-1}q_2(z_0) - \overline{\partial}^{-1}q_2)\right) + e^{i\tau\overline{\Phi}}r_2$$

Using these will give as $\tau \to \infty$

$$0 = \int (q_1 - q_2)u_1 u_2 = \text{main term} + \text{cross terms} + O(\tau^{-1/3})$$

The main term $\to (q_1 - q_2)(z_0)$. The cross terms behave as follows

$$\begin{aligned}
&\frac{2\tau}{\pi}\int \partial^{-1}\left(e^{-i\tau(\Phi+\overline{\Phi})}(\overline{\partial}^{-1}q_2(z_0) - \overline{\partial}^{-1}q_2)\right)(q_1 - q_2)dm \\
&\quad = -\frac{2\tau}{\pi}\int e^{-i\tau(\Phi+\overline{\Phi})}(\overline{\partial}^{-1}q_2(z_0) - \overline{\partial}^{-1}q_2)\partial^{-1}(q_1 - q_2)dm \\
&\qquad \longrightarrow (\overline{\partial}^{-1}q_2(z_0) - \overline{\partial}^{-1}q_2(z_0))\partial^{-1}(q_1 - q_2)(z_0) = 0 \quad \square
\end{aligned}$$

Explanation 2: why u_0 instead of $\overline{\partial}^{-1}(e^{-i\tau(\Phi+\overline{\Phi})})$

$$\overline{\partial}^{-1}(e^{-i\tau(\Phi+\overline{\Phi})})(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{e^{-i\tau(\Phi+\overline{\Phi})}}{z-z'}dm(z') \qquad ???$$

Theorem

Let $\tau > 1$ and $z_0 \in \mathbb{C}$. Then there is $u_0 \in L^\infty(\mathbb{R}^2)$ such that

$$\overline{\partial}u_0 = e^{-i\tau(\Phi+\overline{\Phi})}.$$

Moreover if $Q \in L^1 \cap L^{(2,1)}(\mathbb{R}^2)$ then

$$\lim_{\tau\to\infty} \frac{2\tau}{\pi}\int_{\mathbb{C}} Q(z)u_0(z)dm(z) = -\overline{\partial}^{-1}Q(z_0) \quad \text{in } \mathscr{D}'(\mathbb{R}^2).$$

Proof.

If $\Psi \in C_0^\infty(\mathbb{R}^2)$, $\Psi \equiv 1$ near 0, $H(z) = \frac{1-\Psi(z-z_0)}{\overline{z}-\overline{z_0}}$ then define

$$u_0 = \overline{\partial}^{-1}\left(e^{-i\tau(\Phi+\overline{\Phi})}\Psi(\cdot - z_0)\right) - \frac{1}{2i\tau}\left(e^{-i\tau(\Phi+\overline{\Phi})}H - \overline{\partial}^{-1}(e^{-i\tau(\Phi+\overline{\Phi})}\overline{\partial}H)\right)$$

$\square$

Thank you for your attention!