Solving the Inverse Problem for the 2D Schrödinger Equation with L^p-potential

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Joint work with Oleg Imanuvilov and Masahiro Yamamoto

Inverse problems for partial differential equations

The Calderón problem: given an open set $\Omega \subset \mathbb{R}^n$ and all (voltage, current flux) pairs $(v, f) \in H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ satisfying

$$\begin{aligned} \nabla \cdot \gamma \nabla u &= 0 \quad & \Omega \\ u &= v \quad & \partial\Omega \\ \gamma \partial_\nu u &= f \quad & \partial\Omega \end{aligned}$$

deduce the conductivity γ inside Ω. Asked by Calderón and he solved the linearised problem.
Another well-studied problem is inverse scattering: given

$$\mathcal{C}_q = \left\{ (u_{|\partial\Omega}, \partial_\nu u_{|\partial\Omega}) \,\middle|\, (\Delta + q)u = 0, u \in H^1(\Omega) \right\},$$

deduce the scattering potential q inside Ω.

Purpose of the talk

- Situation related to nonsmooth potentials in 2D?
- How the inverse problem is solved in 2D?
- Status of our joint work with Imanuvilov and Yamamoto?

Some papers before Bukhgeim

- Calderón 1980 (manuscript from 60's): linearised problem
- Kohn & Vogelius 1984: piecewise analytic γ
- Sylvester & Uhlmann 1987: general smooth γ and/or q in 3D
- Alessandrini 1988: logarithmic stability result
- Astala & Päivärinta 2006: general $\gamma \in L^\infty$ in 2D
- Bukhgeim 2008: Schrödinger equation with $q \in W^{1,p}$ in 2D
- Novikov & Santacesaria 2010: stability for $q \in C^2$ in 2D

Also, among others: Nachman, Liu, Jerison, Kenig, ...

Complex Geometric Optics solutions

$$u(x) = e^{\rho \cdot x}(1+\varepsilon), \quad \rho \in \mathbb{C}^n, \quad \rho \cdot \rho = 0,$$

or

$$u(z) = e^{i\tau\Phi(z)}(1+\varepsilon), \quad \overline{\partial}\Phi = 0, \quad \tau \gg 1.$$

What happened regarding Bukhgeim's paper: my point of view

- 2010 - licentiate thesis
- 2012 - Imanuvilov & Yamamoto: uniqueness for $q \in L^p$, in arXiv
- 2013 - doctoral thesis: stability for $q \in W_p^{\varepsilon}$

Then meanwhile Albin, Guillarmou, Tzou, Uhlmann, Lai, Wang, Salo, Barceló, Clop, Astala, Faraco, Rogers, Ruíz, Imanuvilov, Yamamoto, Novikov, Santacesaria and others generalize the 2D case: reconstruction, partial boudary data results, scattering with noncompact support, systems, many kinds of elliptic PDE's

The focus of our research: reduce smoothness assumptions!

Our main results

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and $p > 2$.

Theorem (Uniqueness)

Assume that $q_1, q_2 \in L^p(\Omega)$ with $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$. Then $q_1 = q_2$.

Theorem (Stability)

Let $\varepsilon > 0$ and $M < \infty$. Then there exists constants $C, d_0, \theta > 0$ such that

$$\|q_1 - q_2\|_{L^2(\Omega)} \leq C \left(\ln \frac{1}{d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})} \right)^{-\theta}$$

if $q_1, q_2 \in W_p^{\varepsilon}(\Omega)$ with norms at most M and $d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) \leq d_0$.

Three ingredients of the proofs

By expanding by zero

$$(q_1-q_2)(z_0) = (q_1-q_2)(z_0) - \int_{\mathbb{R}^2} \tau e^{i\tau(\Phi(z)+\overline{\Phi}(z))}(q_1-q_2)(z)dm(z)$$
$$+ \tau \int_\Omega (q_1 - q_2) u_1 u_2(z) dm(z)$$
$$- \tau \int_\Omega (q_1 - q_2)(z)\big(u_1 u_2(z) - e^{i\tau(\Phi(z)+\overline{\Phi}(z))}\big) dm(z).$$

For the proof we will have u_1 and u_2 the CGO-solutions to $(\Delta + q_j)u_j = 0$ and Φ a holomorphic Morse function. We will let $\tau \to \infty$.

Estimating the three ingredients

- Stationary phase, for Φ Morse,

$$\int_{\mathbb{R}^2} \tau e^{i\tau(\Phi(z)+\overline{\Phi}(z))}(q_1-q_2)(z)dm(z) \longrightarrow \sum_{z_0\in\{\text{non-deg. crit. points of }\Phi\}} C_{z_0}(q_1-q_2)(z_0),$$

- Orthogonality relation for admissible solutions $(\Delta + q_j)u_j = 0$

$$\int_\Omega (q_1 - q_2)u_1 u_2 = \text{small and known from boundary data},$$

- Error term integral

$$\left\| \tau \int_\Omega (q_1 - q_2)(z)(u_1 u_2 - e^{i\tau(\Phi(z)+\overline{\Phi}(z))})dm(z) \right\| \leq ?$$

Simple parts

From now on $\Phi(z) := (z - z_0)^2$.

Lemma

Let $Q \in L^2(\Omega)$. Then

$$\lim_{\tau\to\infty} \left\| Q(z_0) - \int_{\mathbb{R}^2} \tau e^{i\tau((z-z_0)^2+(\overline{z}-\overline{z_0})^2)} Q(z)dm(z) \right\|_{L^2(\Omega)} = 0.$$

If $Q \in W_2^\varepsilon(\Omega)$ with $\varepsilon < 1/2$, then the norm is at most $C\tau^{-\varepsilon/2}$ for each τ.

Lemma

For any admissible solutions u_1, u_2 to $(\Delta + q_j)u_j = 0$ we have

$$\left| \tau \int_\Omega (q_1 - q_2)u_1 u_2 dm \right| \leq \tau \|u_1\|_{W_2^1(\Omega)} \|u_2\|_{W_2^1(\Omega)} d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}).$$

The third part (for stability)

If

$$\begin{array}{l} u_1(z) = e^{i\tau(z-z_0)^2}(1+r_1(z)) \\ u_2(z) = e^{i\tau(\overline{z}-\overline{z_0})^2}(1+r_2(z)) \end{array}, \qquad \mathcal{R} = r_1 + r_2 + r_1 r_2,$$

then the third ingredient, the error integral, is

$$\tau \int_\Omega e^{i\tau((z-z_0)^2+(\overline{z}-\overline{z_0})^2)}(q_1-q_2)(z)\mathcal{R}(z)dm(z) =: \mathcal{I}_\tau(z_0)$$

- $q_1, q_2 \in L^p(\Omega) \implies |\mathcal{I}_\tau(z_0)| \le C\tau \|q_1 - q_2\|_{L^p(\Omega)} \|\mathcal{R}\|_{L^p(\Omega)}$
- $q_1, q_2 \in W^1_p(\Omega) \implies |\mathcal{I}_\tau(z_0)| \le C \|q_1 - q_2\|_{W^1_p(\Omega)} \|\mathcal{R}\|_{W^1_p(\Omega)}$

By interpolation

$$|\mathcal{I}_\tau(z_0)| \le C\tau^{1-\varepsilon} \|q_1 - q_2\|_{W^\varepsilon_p(\Omega)} \|\mathcal{R}\|_{W^\varepsilon_p(\Omega)}.$$

For stability, show that $\|\mathcal{R}\|$ vanishes fast enough as $\tau \to \infty$!

Existence of $u = e^{i\tau\Phi}(1 + r)$ with enough decay on r?

Because $\Delta = 4\partial\overline{\partial}$,

$$\begin{cases} \overline{\partial} f = e^{-i\tau(\Phi+\overline{\Phi})} g \\ \partial g = -\frac{1}{4} q e^{i\tau(\Phi+\overline{\Phi})} f \end{cases} \Longrightarrow (\Delta + q)(e^{i\tau\Phi} f) = 0.$$

Hence it is enough to consider the integral equation

$$f = 1 - \tfrac{1}{4}\overline{\partial}^{-1}\left(e^{-i\tau(\Phi+\overline{\Phi})}\partial^{-1}\left(e^{i\tau(\Phi+\overline{\Phi})} q f\right)\right)$$

where

$$\overline{\partial}^{-1} h(z) = \frac{1}{\pi}\int \frac{h(z')}{z - z'} dm(z'), \qquad \partial^{-1} h(z) = \frac{1}{\pi}\int \frac{h(z')}{\overline{z} - \overline{z'}} dm(z').$$

Existence, Carleman estimate

Mapping properties:

$$\overline{\partial}^{-1}, \partial^{-1} : L^{p^*}(\Omega) \to L^p(\Omega), \quad L^p(\Omega) \to L^\infty(\Omega)$$

for

$$2 < p < \infty, \quad 1 < p^* < 2, \quad \frac{1}{p^*} = \frac{1}{2} + \frac{1}{p}.$$

Theorem

Let $\chi \in C_0^\infty(\mathbb{R}^2)$. Then

$$\left\|\partial^{-1}(e^{i\tau(\Phi+\overline{\Phi})}\chi a)\right\|_{L^p(\mathbb{R}^2)} \leq C\tau^{-1/2-1/p}\|a\|_{W_p^1(\mathbb{R}^2)},$$

$$\left\|\partial^{-1}(e^{i\tau(\Phi+\overline{\Phi})}\chi a)\right\|_{L^\infty(\mathbb{R}^2)} \leq C\tau^{-1/p}\|a\|_{W_p^1(\mathbb{R}^2)}.$$

Existence of u

Theorem

Let $q \in W_p^{\varepsilon}(\Omega)$. *If* $\tau > \tau_0$, *there is* $u \in W_2^1(\mathbb{R}^2)$ *such that* $\Delta u + qu = 0$ *in* Ω *and*

$$u(z) = e^{i\tau\Phi}(1 + r(z)),$$

$$\sup_{z_0} \|r\|_{W_p^{\varepsilon}(\Omega)} \leq C\tau^{-1/2-1/p}.$$

Stability

Now estimate all terms in $L^2(\Omega)$ from

$$
\begin{aligned}
(q_1 - q_2)(z_0) = (q_1 - q_2)(z_0) &- \int_{\mathbb{R}^2} \tau e^{i\tau(\Phi(z)+\overline{\Phi}(z))}(q_1 - q_2)(z)dm(z) \\
&+ \tau \int_{\Omega} (q_1 - q_2) u_1 u_2(z) dm(z) \\
&- \tau \int_{\Omega} (q_1 - q_2)(z)\big(u_1 u_2(z) - e^{i\tau(\Phi(z)+\overline{\Phi}(z))}\big) dm(z).
\end{aligned}
$$

Stability

Now estimate all terms in $L^2(\Omega)$ from

$$
\begin{aligned}
(q_1 - q_2)(z_0) &= (q_1 - q_2)(z_0) - \int_{\mathbb{R}^2} \tau e^{i\tau(\Phi(z) + \overline{\Phi}(z))}(q_1 - q_2)(z)dm(z) \\
&\quad + \tau \int_\Omega (q_1 - q_2)u_1 u_2(z)dm(z) \\
&\quad - \tau \int_\Omega e^{i\tau((z-z_0)^2 + (\overline{z} - \overline{z_0})^2)}(q_1 - q_2)(z)\mathcal{R}(z)dm(z).
\end{aligned}
$$

Stability

Now estimate all terms in $L^2(\Omega)$ from

$$
\begin{aligned}
\|q_1 - q_2\|_{L^2} \leq & \left\| (q_1 - q_2) - \int_{\mathbb{R}^2} \tau e^{i\tau(\Phi(z)+\overline{\Phi}(z))}(q_1 - q_2)(z)\,dm(z) \right\|_{L^2} \\
& + \tau \left\| \int_\Omega (q_1 - q_2) u_1 u_2(z)\,dm(z) \right\|_{L^2} \\
& + \tau \left\| \int_\Omega e^{i\tau((z-z_0)^2+(\overline{z}-\overline{z_0})^2)}(q_1 - q_2)(z)\mathcal{R}(z)\,dm(z) \right\|_{L^2}.
\end{aligned}
$$

- stationary phase

Stability

Now estimate all terms in $L^2(\Omega)$ from

$$
\begin{aligned}
\|q_1 - q_2\|_{L^2} \leq C\tau^{-\varepsilon/2} \\
&+ \tau \Big\| \int_\Omega (q_1 - q_2) u_1 u_2(z) dm(z) \Big\|_{L^2} \\
&+ \tau \Big\| \int_\Omega e^{i\tau((z-z_0)^2 + (\overline{z} - \overline{z_0})^2)} (q_1 - q_2)(z) \mathcal{R}(z) dm(z) \Big\|_{L^2}.
\end{aligned}
$$

- stationary phase
- boundary data and norms of u_j

Stability

Now estimate all terms in $L^2(\Omega)$ from

$$
\begin{aligned}
\|q_1 - q_2\|_{L^2} \leq\ & C\tau^{-\varepsilon/2} \\
& + C\tau e^{C\tau} d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) \\
& + \tau \left\| \int_\Omega e^{i\tau((z-z_0)^2 + (\overline{z}-\overline{z_0})^2)} (q_1 - q_2)(z) \mathcal{R}(z)\, dm(z) \right\|_{L^2}.
\end{aligned}
$$

- stationary phase
- boundary data and norms of u_j
- error norms given by the Carleman estimate

Stability

Now estimate all terms in $L^2(\Omega)$ from

$$\begin{aligned}\|q_1 - q_2\|_{L^2} \leq\; & C\tau^{-\varepsilon/2} \\ & + C\tau e^{C\tau} d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) \\ & + C\tau^{1-\varepsilon-1/2-1/p}.\end{aligned}$$

- stationary phase
- boundary data and norms of u_j
- error norms given by the Carleman estimate
- p sufficiently close to 2

Stability

Now estimate all terms in $L^2(\Omega)$ from

$$
\begin{aligned}
\|q_1 - q_2\|_{L^2} &\leq C\tau^{-\varepsilon/2} \\
&\quad + C\tau e^{C\tau} d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) \\
&\quad + C\tau^{-\varepsilon/2}.
\end{aligned}
$$

- stationary phase
- boundary data and norms of u_j
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Stability

Now estimate all terms in $L^2(\Omega)$ from

$$\begin{aligned}\|q_1 - q_2\|_{L^2} &\leq C\tau^{-\varepsilon/2} \\ &\quad + C\tau e^{C\tau} d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) \\ &\quad + C\tau^{-\varepsilon/2}.\end{aligned}$$

- stationary phase
- boundary data and norms of u_j
- error norms given by the Carleman estimate
- p sufficiently close to 2

Implies $\|q_1 - q_2\|_{L^2(\Omega)} \leq C\left(\ln \frac{1}{d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})}\right)^{-\theta}$ for a choice of τ.

How is uniqueness shown?

Actually:

$$(\Delta + q)(e^{i\tau\Phi} f) = 0 \iff \begin{cases} \overline{\partial} f = e^{-i\tau(\Phi+\overline{\Phi})} g \\ \partial g = -\frac{1}{4} q e^{i\tau(\Phi+\overline{\Phi})} f \end{cases}$$

so one may consider

$$\begin{cases} f = \psi_1 + \overline{\partial}^{-1}(e^{-i\tau(\Phi+\overline{\Phi})} g), & \overline{\partial}\psi_1 = 0 \\ g = \psi_2 - \frac{1}{4}\partial^{-1}(q e^{i\tau(\Phi+\overline{\Phi})} f), & \partial\psi_2 = 0 \end{cases}$$

So use new types of solutions than $(\psi_1, \psi_2) = (1, 0)$!

CGO solutions from Imanuvilov and Yamamoto

Solutions to $(\Delta + q)u = 0$ from Imanuvilov and Yamamoto's 2012 manuscript:

$$
\begin{aligned}
u(z) &= e^{i\tau\Phi} f(z) \\
f(z) &= e^{-i\tau(\Phi+\overline{\Phi})} + \tfrac{1}{4}\partial^{-1} q(z_0)\overline{\partial}^{-1}\big(e^{-i\tau(\Phi+\overline{\Phi})}\chi\big) \\
&\qquad\qquad\qquad - \tfrac{1}{4}\overline{\partial}^{-1}\big(e^{-i\tau(\Phi+\overline{\Phi})}\chi\partial^{-1}(e^{i\tau(\Phi+\overline{\Phi})} q f)\big) \\
&= e^{-i\tau(\Phi+\overline{\Phi})} + \tfrac{1}{4}\overline{\partial}^{-1}\big(e^{-i\tau(\Phi+\overline{\Phi})}(\partial^{-1} q(z_0) - \partial^{-1} q)\chi\big) + r.
\end{aligned}
$$

Now there is a better decay on the error

$$\sup_{z_0} \|r\|_{L^p(\Omega)} \leq C\tau^{-1/2-\mathbf{2}/p}.$$

Uniqueness

By

$$u_1 = e^{-i\tau\overline{\Phi}} + \frac{1}{4} e^{i\tau\Phi} \overline{\partial}^{-1} \big(e^{-i\tau(\Phi+\overline{\Phi})} (\partial^{-1} q_1(z_0) - \partial^{-1} q_1)\chi \big) + e^{i\tau\Phi} r_1$$
$$u_2 = e^{-i\tau\Phi} + \frac{1}{4} e^{i\tau\overline{\Phi}} \partial^{-1} \big(e^{-i\tau(\Phi+\overline{\Phi})} (\overline{\partial}^{-1} q_2(z_0) - \overline{\partial}^{-1} q_2)\chi \big) + e^{i\tau\overline{\Phi}} r_2$$

and the equality of the Cauchy data

$$0 = \tau \int (q_1 - q_2) u_1 u_2 dm(z)$$

Uniqueness

By

$$u_1 = e^{-i\tau\overline{\Phi}} + \frac{1}{4}e^{i\tau\Phi}\overline{\partial}^{-1}\left(e^{-i\tau(\Phi+\overline{\Phi})}(\partial^{-1}q_1(z_0) - \partial^{-1}q_1)\chi\right) + e^{i\tau\Phi}r_1$$
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and the equality of the Cauchy data

$$0 = \tau\int(q_1 - q_2)u_1u_2dm(z) = \tau\int e^{-i\tau(\Phi+\overline{\Phi})}(q_1 - q_2)dm(z)$$

Uniqueness

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$$u_1 = e^{-i\tau\overline{\Phi}} + \frac{1}{4}e^{i\tau\Phi}\overline{\partial}^{-1}\left(e^{-i\tau(\Phi+\overline{\Phi})}(\partial^{-1}q_1(z_0) - \partial^{-1}q_1)\chi\right) + e^{i\tau\Phi}r_1$$

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$$+\frac{\tau}{4}\int\overline{\partial}^{-1}\left(e^{-i\tau(\Phi+\overline{\Phi})}(\partial^{-1}q_1(z_0) - \partial^{-1}q_1)\chi\right)(q_1 - q_2)(z)dm(z)$$

Uniqueness

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$$0 = \tau\int(q_1 - q_2)u_1u_2\,dm(z) = \tau\int e^{-i\tau(\Phi+\overline{\Phi})}(q_1 - q_2)\,dm(z)$$
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$$+\ \text{similar for } q_2$$

Uniqueness

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$$0 = \tau \int (q_1 - q_2) u_1 u_2 dm(z) = \tau \int e^{-i\tau(\Phi+\overline{\Phi})}(q_1 - q_2) dm(z)$$
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$$+ \text{ similar for } q_2 + \mathcal{O}(\tau^{1/2-2/p}).$$

Uniqueness

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$$+\ \text{similar for } q_2 + \mathcal{O}(\tau^{1/2-2/p}).$$

For the middle term: $\int v\overline{\partial}^{-1}wdm = \int w\overline{\partial}^{-1}vdm$

Uniqueness

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$$+\frac{\tau}{4}\int e^{-i\tau(\Phi+\overline{\Phi})}(\partial^{-1}q_1(z_0) - \partial^{-1}q_1(z))\chi(z)\partial^{-1}(q_1 - q_2)(z)dm(z)$$
$$+ \text{ similar for } q_2 + \mathcal{O}(\tau^{1/2-2/p}).$$

Letting $\tau \to \infty$ we get $0 = (q_1 - q_2)(z_0)$.

Future work in 2D

- q of unbounded support with Gaussian decay and no smoothness
- connection between PDE's and first order systems, especially with respect to boundary data

¡Thank you for your attention!