Completeness of the generalized transmission eigenstates

Eemeli Blåsten

Department of Mathematics and Statistics, University of Helsinki
Research carried out at the Mittag-Leffler Institute

Joint work with Lassi Päivärinta

The interior transmission problem

The interior transmission problem (of the Helmholtz equation) is the following degenerate boundary value problem,

$$\begin{aligned} (\Delta - \lambda)v &= 0 \quad \text{in} \quad \Omega, \\ (\Delta - \lambda(1+V))w &= 0 \quad \text{in} \quad \Omega, \\ v - w &\in H_0^2(\Omega). \end{aligned} \tag{ITP}$$

We say that $\lambda \in \mathbb{C}$ is a transmission eigenvalue (TE) if (ITP) has non-trivial solutions $0 \neq v \in L^2_{\mathrm{loc}}$ and $0 \neq w \in L^2_{\mathrm{loc}}$.

ITP in the Schrödinger case

The interior transmission problem of the Schrödinger equation is the following degenerate boundary value problem,

$$\begin{aligned} (\Delta - \lambda)v = 0 \quad &\text{in} \quad \Omega, \\ (\Delta - \lambda + V)w = 0 \quad &\text{in} \quad \Omega, \\ v - w \in H_0^2(\Omega). \end{aligned}$$

Transmission eigenvalues can be defined for many different kind of boundary value problem. The method presented deals with both the Helmholtz and the Schrödinger cases simultaneously, among others.

History

- Introduction of ITP: Colton and Monk, 1988, in connection with an inverse scattering problem.
- Discreteness of the set of TE's: Colton, Kirsch and Päivärinta 1989.
- Existence of real TE's (Päivärinta and Sylvester 2008).
- Existence of an infinite set of real TE's (Cakoni, Gintides, and Haddar 2010).
- Existence of complex TE's, assuming that V is constant sufficiently close to 0 (Cakoni, Colton and Gintides 2010).

Various equations and their TE's studied:

- TE's for Maxwell's equations (Kirsch 2007, Cakoni and Kirsch 2010).
- TE's for the Helmholtz equation in the presence of cavities (Cakoni, Colton and Haddar, 2010).
- TE's for constant coefficient operators (Hitrik, Krupchyk, Ola and Päivärinta 2010)

Our goal

Cakoni, Gintides, Haddar 2010: "We think that some interesting open problems [are] . . . , . . . and the completeness of the eigensystem of the interior transmission problem."

We shall prove the completeness of a system of generalized eigenstates.

Characterization of TE's

Let $v, w \neq 0$ be solutions to

$$\begin{aligned}(\Delta - \lambda)v = 0 \quad &\text{in} \quad \Omega,\\ (\Delta - \lambda(1+V))w = 0 \quad &\text{in} \quad \Omega,\\ v - w \in H_0^2(\Omega).&\end{aligned}$$

Assume that V is real-valued, smooth and such that $V \geq \delta > 0$ a.e. in $\overline{\Omega}$.

Then $u := w - v$ satisfies

$$u \in H^4 \cap H_0^2(\Omega),$$
$$T(\lambda)u := (\Delta - \lambda(1+V))\frac{1}{V}(\Delta - \lambda)u = 0.$$

Reduction to a higher-order eigenvalue problem

The question of deciding whether $0 \neq \lambda \in \mathbb{C}$ is a TE is equivalent to finding a non-trivial solution $u \in H_0^2(\Omega)$ of the following quadratic eigenvalue problem

$$T(\lambda)u = (A_0 + A_1\lambda + A_2\lambda^2)u = 0,$$

$$A_0 = \Delta\frac{1}{V}\Delta, \quad A_1 = -\frac{1}{V}\Delta - \Delta\frac{1}{V} - \Delta, \quad A_2 = 1 + \frac{1}{V}.$$

Generalized transmission eigenstates

Here B_0, B_1 and B_2 are Taylor coefficients:

$$T(\lambda) = B_0 + B_1(\lambda - \lambda_0) + B_2(\lambda - \lambda_0)^2,$$
$$B_0 = T(\lambda_0), \quad B_1 = \tfrac{d}{d\lambda} T(\lambda_0), \quad B_2 = \tfrac{1}{2}\tfrac{d^2}{d\lambda^2} T(\lambda_0).$$

Let λ_0 be a TE. The generalized eigenspace $\mathcal{E}_{\lambda_0}$ is the closed linear space spanned by the vectors $(u_j)_{j=0}^{\infty}$, $u_j \in H_0^2(\Omega)$, where

$$\begin{aligned} &B_0 u_0 = 0, \qquad u_0 \neq 0, \\ &B_1 u_0 + B_0 u_1 = 0, \\ &B_2 u_{j-2} + B_1 u_{j-1} + B_0 u_j = 0, \quad j = 2, 3, \ldots \end{aligned}$$

Another definition of generalized eigenstates

A vector w is called a generalized eigenvector of a matrix M if

$$(M - \lambda_0)^k w = 0$$

for some eigenvalue λ_0 and $k \in \mathbb{N}$.

Relationship between the two definitions

Consider

$$T(\lambda) = A_0 + A_1\lambda + A_2\lambda^2 = B_0 + B_1(\lambda-\lambda_0) + B_2(\lambda-\lambda_0)^2, \quad B_2 \text{ invertible.}$$

Now

$$\begin{aligned} &B_0 u_0 = 0, \qquad u_0 \neq 0, \\ &B_1 u_0 + B_0 u_1 = 0, \\ &B_2 u_{j-2} + B_1 u_{j-1} + B_0 u_j = 0, \quad j = 2, 3, \ldots \end{aligned}$$

if and only if

$$(\mathcal{A} - \lambda_0)^{j+1} \begin{pmatrix} u_j \\ v_j \end{pmatrix} := 0, \quad \text{where } \mathcal{A} = \begin{pmatrix} 0 & A_2^{-1} \\ -A_0 & -A_1 A_2^{-1} \end{pmatrix}.$$

for $j = 0, 1, \ldots$

Completeness result

Theorem

Assume that $V \in C^\infty(\overline{\Omega})$ *and* $V > 0$ *on* $\overline{\Omega}$. *Then the space* $\bigoplus_{\lambda \in \mathbb{C}} \mathcal{E}_\lambda$ *is complete in* $L^2(\Omega)$. *This is true for both the Helmholtz and Schrödinger ITP.*

We will use

- the analytic Fredholm theorem
- generalized Shapiro-Lopatinsky conditions by Agranovich and Vishik (1964) to invert $T(\lambda)$
- Nevanlinna theory to estimate $\| T(\lambda)^{-1} \|$

Some history of eigenstate completeness results

- ...
- Keldysh (1950's)
- Agranovich (1978)
- Robert and Lai (1980)
- Robbiano (2013)

Key-ingredient in eigenstate completeness proofs

Theorem (Phragmén-Lindelöf)

Let w be a holomorphic function in a sector $\{\lambda \in \mathbb{C} \mid \alpha_1 < \arg \lambda < \alpha_2\}$. Let $|w| \leq 1$ on its boundary and

$$\sup_{|\lambda|=R_j} |w(\lambda)| \leq \exp R_j^\beta, \qquad \beta < \frac{\pi}{\alpha_2 - \alpha_1}$$

for a sequence $R_1, R_2, \ldots \to \infty$. Then $|w(\lambda)| \leq 1$ in the whole sector.

Three sufficient assumption

The space $\mathcal{E}_\lambda$ can be defined for analytic families of operators too.

Theorem

Let H be a Hilbert spaces and $D \hookrightarrow H$ a dense normed subspace. Assume that

1. $T : \mathbb{C} \to \mathscr{L}(D, H)$ *is analytic*
2. $T(\lambda)^{-1}$ *is bounded on certain rays and* $\|T(\lambda_j)^{-1}\| \to 0$ *on some sequence* λ_j
3. $T(\lambda)^{-1}$ *is meromorphic* $\mathbb{C} \to \mathscr{L}(H, H)$ *of finite order*

Then $\bigoplus_{\lambda \in \mathbb{C}} \mathcal{E}_\lambda$ *is complete in* H.

Meromorphic of finite order?

An entire $\varphi : \mathbb{C} \to B$ is of order p if

$$\|\varphi(\lambda)\|_B \leq \exp C_\varepsilon |\lambda|^{p+\varepsilon}$$

for all $\varepsilon > 0$ and $|\lambda| > 1$.

A *meromorphic* $\omega : \mathbb{C} \to B$ is of order p if

$$\omega(\lambda) = \frac{\omega_1(\lambda)}{\omega_2(\lambda)}, \qquad \omega_1 \text{ and } \omega_2 \text{ entire of order } p.$$

Proof of completeness, part 1/3

Let λ_0 be a TE, that is, a pole of $T(\lambda)^{-1}$. Write

$$T(\lambda) = \sum_0^\infty B_m (\lambda - \lambda_0)^m, \qquad T(\lambda)^{-1} = \sum_{-N}^\infty C_n (\lambda - \lambda_0)^n$$

and note that

$$\mathrm{Id} = T(\lambda) T(\lambda)^{-1} = \ldots = \sum_{k=0}^{N-1} \Big(\sum_{j=0}^{k} B_{k-j} C_{j-N} \Big) (\lambda - \lambda_0)^{k-N} + \text{analytic part}$$

The left-hand side is analytic, so $\sum_{j=0}^k B_{k-j} C_{j-N} = 0$.

Proof of completeness, part 2/3

For all $v \in H$ we had

$$\sum_{j=0}^{k} B_{k-j} C_{j-N} v = 0.$$

The space $\mathcal{E}_{\lambda_0}$ is spanned by the vectors $(u_j)_{j=0}^{\infty}$, $u_j \in D$, where

$$\begin{aligned} &B_0 u_0 = 0, \qquad u_0 \neq 0, \\ &B_1 u_0 + B_0 u_1 = 0, \\ &\qquad\vdots \\ &\sum_{j=0}^{k} B_{k-j} u_j = 0, \quad k = 2, 3, \ldots \end{aligned}$$

Hence $C_n : H \to \mathcal{E}_{\lambda_0}$ for $n = -1, -2, \ldots, -N$.

Proof of completeness, part 3/3

Let $g \perp \bigoplus_{\lambda \in \mathbb{C}} \mathcal{E}_\lambda$ and $f \in H$ arbitrary, so $T(\lambda)^{-1} f$ is an arbitrary element of D, a dense subset of H.

Near any pole λ_0,

$$w(\lambda) := \left(T(\lambda)^{-1} f, g\right) = \sum_{n=-N}^{\infty} (C_n f, g)(\lambda - \lambda_0)^n = \sum_{n=0}^{\infty} (C_n f, g)(\lambda - \lambda_0)^n,$$

because $C_n : H \to \mathcal{E}_{\lambda_0} \perp g$ for $n < 0$. Hence w is an entire function. By assumptions 2 and 3 and the Phragmén-Lindelöf principle $w(\lambda) \equiv 0$.

Hence $g \perp D$, which is dense, so $g = 0$.

Principal symbol with parameter

In our case,

$$T_H(\lambda) = \Delta\frac{1}{V}\Delta - (\frac{1}{V}\Delta + \Delta\frac{1}{V} + \Delta)\lambda + (1 + \frac{1}{V})\lambda^2$$
$$T_S(\lambda) = \Delta\frac{1}{V}\Delta + \Delta - (\frac{1}{V}\Delta + \Delta\frac{1}{V} + 1)\lambda + \frac{1}{V}\lambda^2$$

Their parameter-dependent principal symbols with weight 2 are

$$T_{H0}(x,\xi,\lambda) = \frac{1}{V}|\xi|^4 + (1 + \frac{2}{V})\lambda|\xi|^2 + (1 + \frac{1}{V})\lambda^2$$
$$T_{S0}(x,\xi,\lambda) = \frac{1}{V}(|\xi|^2 + \lambda)^2$$

The symbols related to the boundary conditions of H_0^2 are

$$B_{00} = 1, \qquad B_{10} = i\nu(x)\cdot\xi.$$

Assumption 2 is satisfied

Generalized Shapiro-Lopatinsky conditions allow us to invert $T(\lambda)$:

Theorem (Agranovich, Vishik, 1964)

Let $Q \in \mathbb{C}$ be a closed cone with vertex at 0. Assume that

- $T_0(x, \xi, \lambda) \neq 0$ *for* $x \in \Omega$ *and* $(\xi, \lambda) \in \mathbb{R}^n \times Q \setminus \{0, 0\}$
- *An ODE coming from the symbols T_0, B_{00} and B_{10} is required to have a unique solution in the set $C_0(\mathbb{R}_+)$.*

Then $T(\lambda) : H^4 \cap H_0^2(\Omega) \to L^2(\Omega)$ is invertible in Q when $|\lambda| > C_Q$, and

$$\|u\|_{H^4} + |\lambda|^2 \|u\|_{L^2} \leq C \| T(\lambda) u \|_{L^2}.$$

$T(\lambda)^{-1}$ is then meromorphic by the analytic Fredholm theorem.

Assumption 3: Meromorphic of finite order

Proposition

$T(\lambda)^{-1}$ is meromorphic of finite order.

Proof:

- $(I - T(\lambda)T(\lambda')^{-1})^m$ is entire, of trace-class and of finite order,
- O. Nevanlinna 2000: $T(\lambda')T(\lambda)^{-1}$ is meromorphic of finite order.

Thank you for your attention