Imaging water supply pipes using pressure waves

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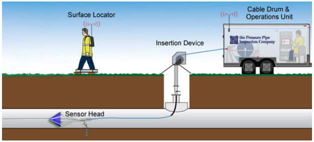
Water supply network

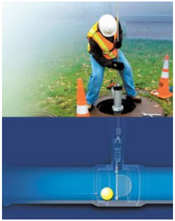
How to locate problems traditionally?

Sahara System



Replace & Rehabilitate

Smart Ball

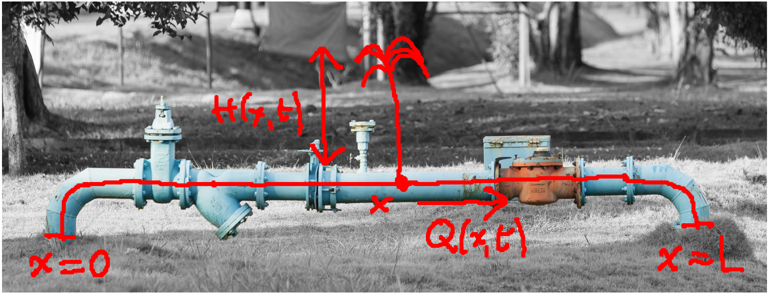


Sonar

Gas Injection

Direct model: single pipe

Direct model: single pipe



$$
\begin{aligned}
\partial_t H + \frac{a^2}{gA}\partial Q &= 0, & 0 < x < L, \quad & t \in \mathbb{R},\\
\partial_t Q + gA\partial H &= 0, & 0 < x < L, \quad & t \in \mathbb{R},\\
H = Q &= 0, & 0 < x < L, \quad & t \leq 0.
\end{aligned}
$$

Water hammer equations.

`https://www.youtube.com/watch?v=jTrhHUwDNYE`

One pipe inverse problem

Measurement $\Lambda(t)$ defined by assuming that H, Q satisfy

$$\begin{aligned} \partial_t H + \frac{a^2}{gA}\partial Q &= 0, \qquad 0 < x < L, \quad t \in \mathbb{R}, \\ \partial_t Q + gA\partial H &= 0, \qquad 0 < x < L, \quad t \in \mathbb{R}, \\ H = Q &= 0, \qquad 0 < x < L, \quad t \leq 0 \end{aligned}$$

Boundary conditions:

whatever static at $x = L$

unit impulse discharge $Q(0,t) = \delta_0(t)$ at $x = 0$

We measure the pressure $\Lambda(t) = H(0,t)$. It gives the impulse-response function.

Recover: $A(x)$, or alternatively $a(x)$, or $A(x)/a(x)$.

Virtual causal solutions

H_v, Q_v are virtual causal solutions if

1. they satisfy the PDE on $0 < x < L$, $t \in \mathbb{R}$,
2. they vanish for $t \leq 0$,
3. they satisfy the boundary condition at $x = L$.

Nothing is required at $x = 0$!

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This is in contrast with the physical solution that gives rise to the measurement Λ.

Integration by parts

For simplicity assume $a(x) = a_0$ constant! Let H_v, Q_v be virtual causal solutions. Then

$$-\partial Q_v = \frac{gA}{a_0^2}\partial_t H_v$$

and integrate $\int_0^\tau \int_0^{a_0\tau} \ldots dxdt$ given any fixed $\tau > 0$:

$$-\int_0^\tau \int_0^{a_0\tau} \partial Q_v(x,t)dxdt = \int_0^\tau \int_0^{a_0\tau} \frac{gA(x)}{a_0^2}\partial_t H_v(x,t)dxdt$$

- $H_v = Q_v = 0$ at $t = 0$
- hence $H_v(x,t) = Q_v(x,t) = 0$ when $x \geqslant a_0 t$, so

$$\int_0^\tau Q_v(0,t)dt = \int_0^{a_0\tau} \frac{gA(x)}{a_0^2} H_v(x,\tau)dx \qquad (1)$$

Special solutions

Given any causal solutions, for example the virtual ones H_v, Q_v, let's look at the total volume input into the system:

$$V(\tau) := \int_0^{\tau} Q_v(0,t)dt = \int_0^{a_0\tau} \frac{gA(x)}{a_0^2} H_v(x,\tau)dx.$$

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then

$$A(x) = \frac{a_0}{g}\frac{\partial V}{\partial \tau}\left(\frac{x}{a_0}\right)$$

After the facts, new problem statement

Unknown: $A(x)$. Known: g, a_0 and physical measurements:

$$Q(0,t) = \delta_0(t), \qquad H(0,t) = \Lambda(t).$$

Given any τ, calculate the boundary values of virtual causal solutions H_v, Q_v for which

$$H_v(x,\tau) = \begin{cases} 1, & x < a_0\tau \\ 0, & x \geqslant a_0\tau \end{cases}.$$

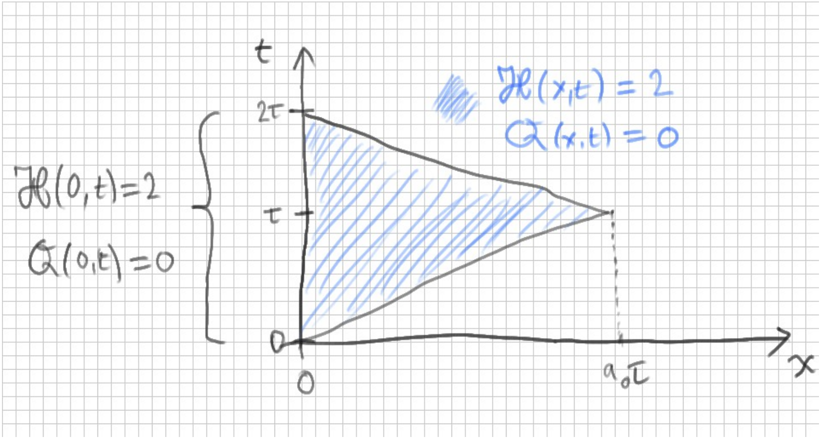
Then $Q_v(0,t)$ for $0 < t < T$ recovers $A(x)$ for $0 < x < T/a_0$.

Unique continuation

If $\mathscr{H}, \mathscr{Q}$ satisfy the PDEs on $0 < x < L$, $0 < t < 2\tau$ and

$$\mathscr{H}(0,t) = 2, \quad \mathscr{Q}(0,t) = 0, \qquad 0 < t < 2\tau,$$

then $\mathscr{H}(x,t) = 2$, $\mathscr{Q}(x,t) = 0$ in $x < a_0(\tau - |\tau - t|)$.



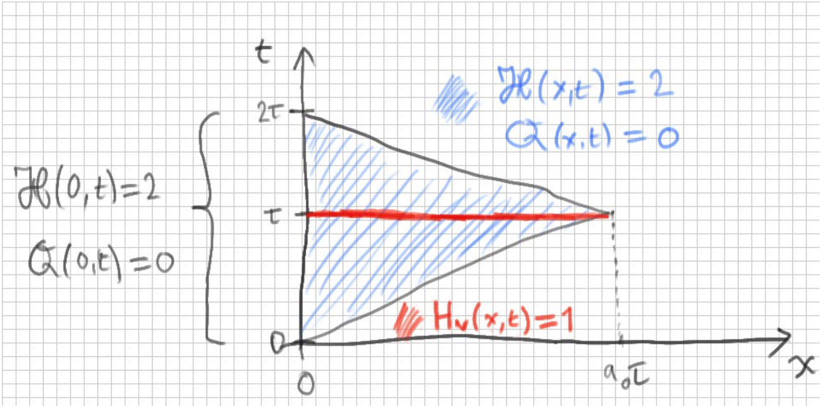
Unique continuation to causal solutions

If H_v, Q_v are virtual causal solutions and $\mathscr{H}, \mathscr{Q}$ defined by

$$\mathscr{H}(x,t) = H_v(x,t) + H_v(x, 2\tau - t), \quad \mathscr{Q}(x,t) = Q_v(x,t) - Q_v(x, 2\tau - t)$$

satisfy $\mathscr{H}(0,t) = 2$, $\mathscr{Q}(0,t) = 0$ on $0 < t < 2\tau$, then

$$H_v(x,\tau) = \tfrac{1}{2}\mathscr{H}(x,\tau) = \begin{cases} 1, & x < a_0\tau \\ 0, & x \geqslant a_0\tau \end{cases}.$$



Next?

Next?

How to find the suitable H_v, Q_v?

Integral equation from requirements of $\mathscr{H}, \mathscr{Q}$

Measurement:

$$Q(0,t) = \delta_0(t), \quad H(0,t) = \Lambda(t) = \frac{a_0}{gA(0)}(\delta_0(t) + h(t))$$

So $Q(0,t) = Q_v(0,t) \implies H(0,t) = \Lambda * Q_v(0,t)$.

Let H_v, Q_v be causal solutions such that $\mathscr{H}(0,t) = 2$, $\mathscr{Q}(0,t) = 0$ on $0 < t < 2\tau$. These imply, in terms of Q_v:

$$Q_v(0,t) + \frac{1}{2}\int_0^{2\tau} Q_v(0,s)h(|s-t|)ds = \frac{gA(0)}{a_0}, \qquad 0 < t < 2\tau.$$

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Conversely, if Q_v solves the above and H_v is the corresponding pressure head, then

$$H_v(x,\tau) = \begin{cases} 1, & x < a_0\tau \\ 0, & x \geqslant a_0\tau \end{cases} \quad \text{at } t = \tau.$$

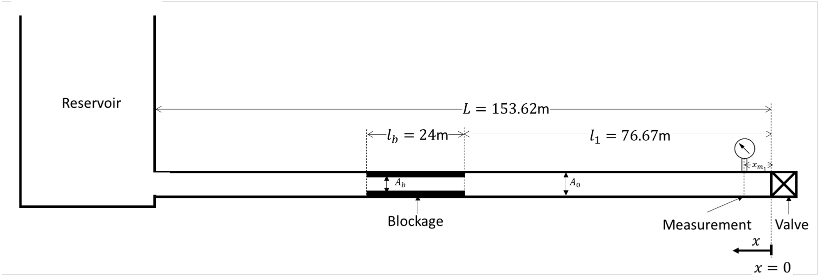
Then $A(a_0\tau) = a_0 g^{-1} \partial_\tau \int_0^\tau Q_v(0,t)dt$.

Algorithm

1. Input $Q(0,t) = \delta_0(t)$ and for $t < 2T = 2L/a_0$ measure
$$H(0,t) = \frac{a_0}{gA(0)}(\delta_0(t) + h(t))$$
2. For whichever $0 < \tau < T$, solve for the boundary value of a virtual causal solution Q_v from
$$Q_v(0,t) + \frac{1}{2}\int_0^{2\tau} Q_v(0,s)h(|s-t|)ds = \frac{gA(0)}{a_0}, \quad 0 < t < 2\tau$$
3. Set
$$V(\tau) = \int_0^{\tau} Q_v(0,t)dt \qquad \left(= \int_0^{a_0\tau} \frac{gA(x)}{a_0^2}dx\right)$$
4. Repeat 2–3 (on the computer) for many τ to get a good approximation of V
5. Given $x < L$ the area can be found by
$$A(x) = \frac{a_0}{g}\left(\frac{\partial}{\partial\tau}V(\tau)\right)_{\tau=x/a_0}$$

Laboratory experiment: setup

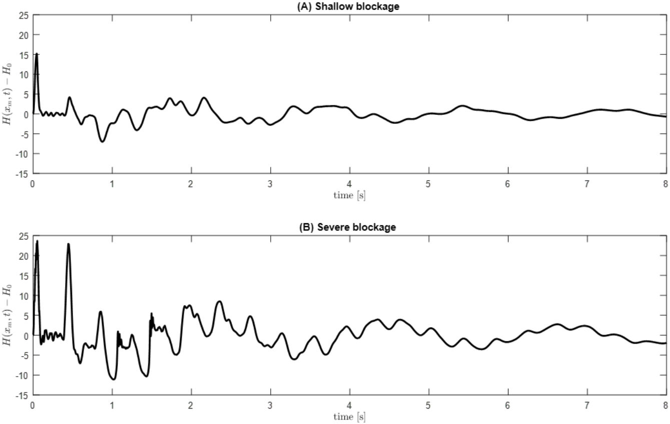
Measurement set up by Silvia Meniconi and Bruno Brunone's group[1].



[1] Università degli Studi di Perugia, Italy

Laboratory experiment: impulse-response function

Measurement set up by Silvia Meniconi and Bruno Brunone's group[2].



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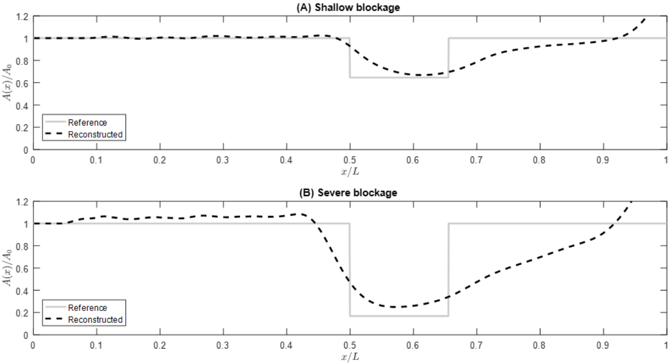
Reconstruction from measured and processed data

Measurement set up by Silvia Meniconi and Bruno Brunone's group[3].



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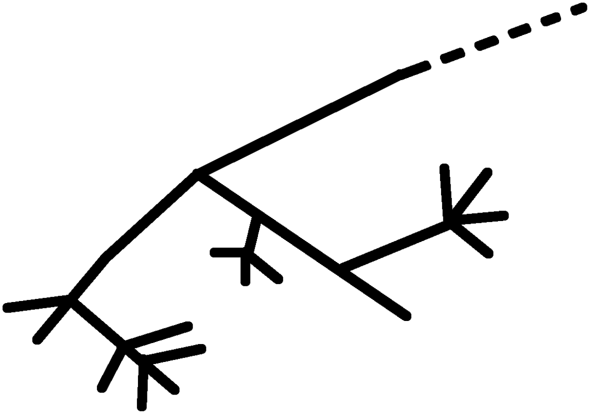
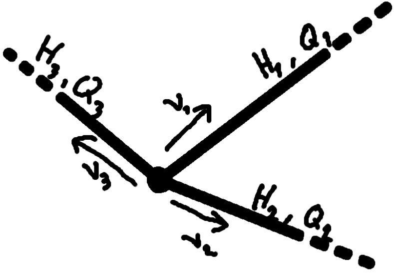
Network



Junction conditions



- H is a scalar: the boundary values H_j are the same at connected pipe ends
- No sinks or sources (total water flowing into pipes sum to zero):

$$\sum_j \nu_j Q_j = 0$$

Main difficulty compared to a segment?

The sets where we can have $H_v(x, \tau) = 1$.

In which Ω can we force $H_v(x, \tau) = 1$?

Control theory suggests that there are boundary values such that a virtual causal solution H_v can be constructed such that

$$H_v(x, \tau) = \begin{cases} 1, & x \in \Omega \\ 0, & x \notin \Omega \end{cases}$$

given any measurable set $\Omega \subset \mathbb{G}$ when $\mathbb{G}$ is a tree and τ large.

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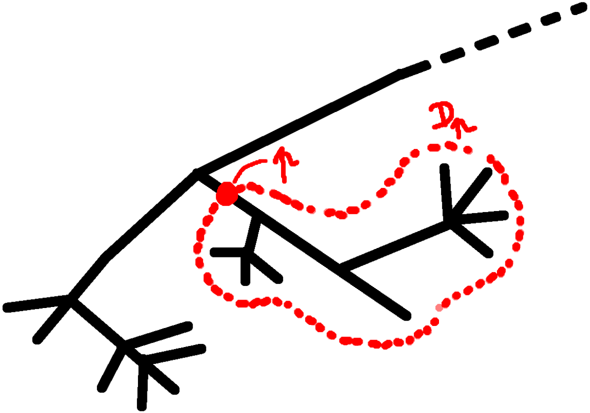
HOWEVER

Can we solve for these boundary values? Is it computationally efficient? Is it even possible?

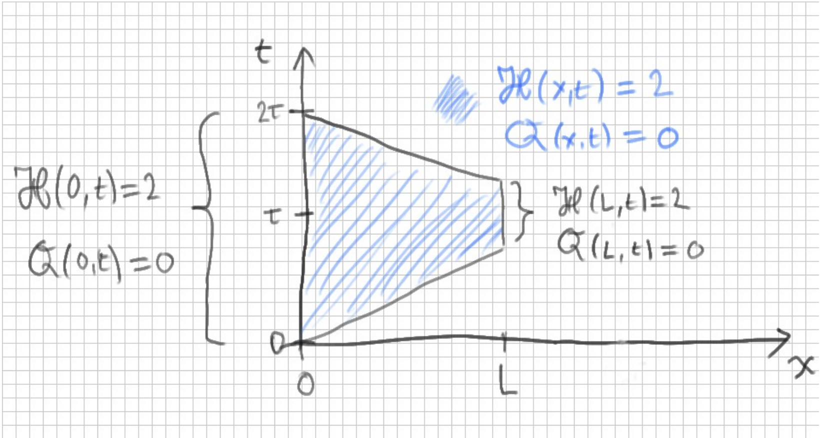
Admissible domains

This works: cut off a branch!



Needs a matrix of measurements!

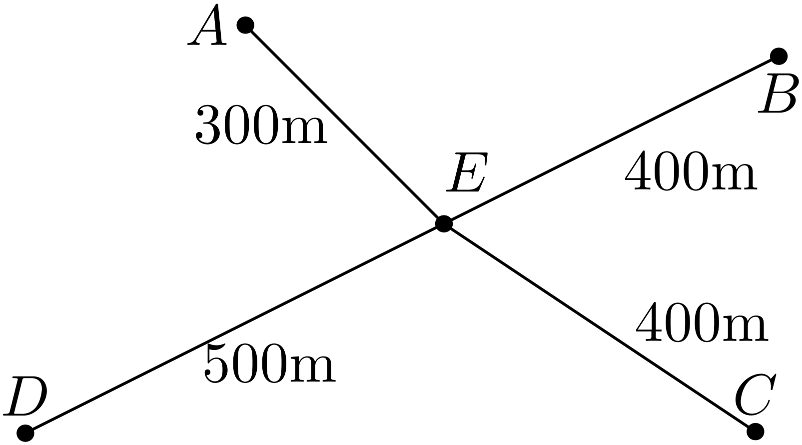
Inductive unique continuation



+ junction conditions
= propagate $\mathscr{H}=2$, $\mathscr{Q}=0$!!

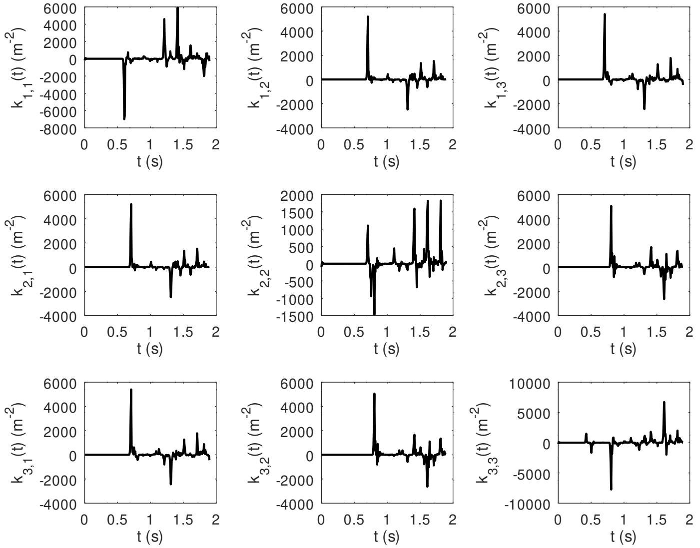
The same logic as before works, but all the equations become more complicated.

Numerical experiment: setup

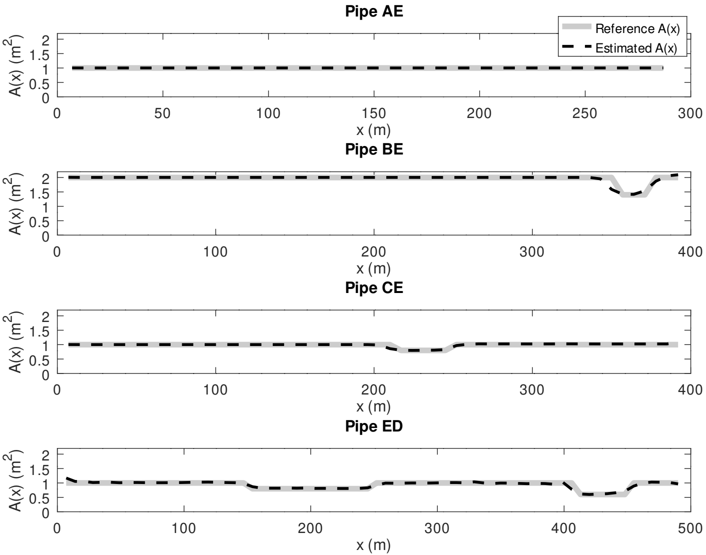


Numerical experiment: impulse-response matrix

$h_{ij}(t) = A(x_j)g/a_0 \cdot k_{ij}(t)$



Reconstruction from measured data using regularization



Danke schön!