Scattering from corners and other singularities

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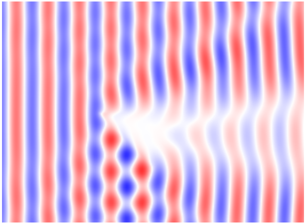
Scattering theory



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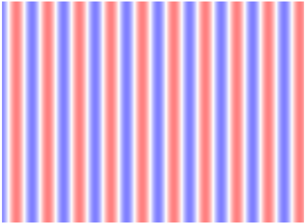

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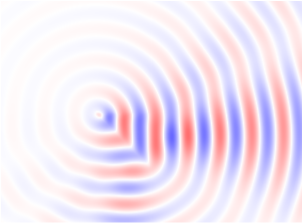
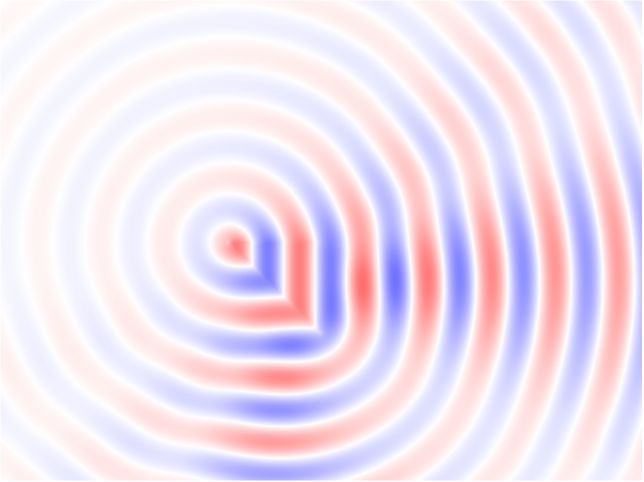


$$u = u^i + u^s$$

Fixed frequency scattering theory: measurements



Measurement: A_{u^i} is the far-field pattern of the scattered wave

$$u^s(x) = \frac{e^{ik|x|}}{|x|^{(n-1)/2}} A_{u^i}\left(\frac{x}{|x|}\right) + \mathcal{O}\left(\frac{1}{|x|^{n/2}}\right)$$

Different inverse scattering problems

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My focus is on single measurement: A_{u^i} given only for a single u^i.

Schiffer's problem: can a single measurement determine Ω?

Why one measurement only?

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Example: Lord Rutherford's gold-foil experiment

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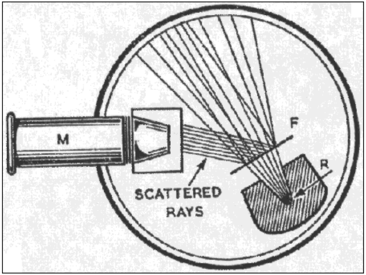

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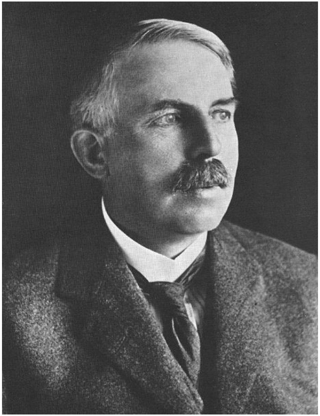

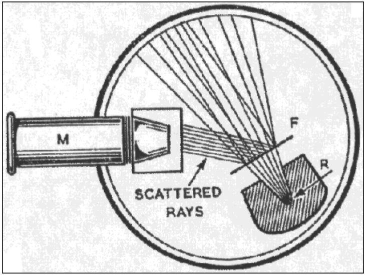

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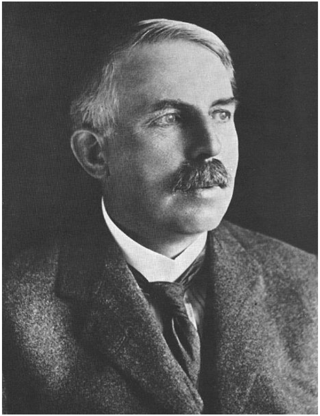
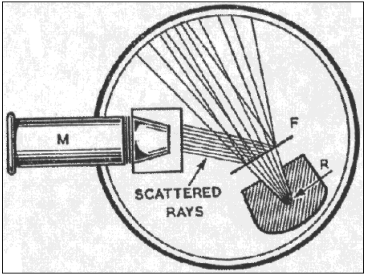
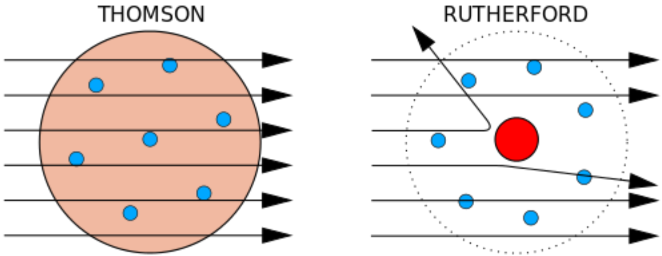




Single incident wave

Scattering theory

Rutherford experiment's conclusions



measurement + a-priori information = conclusion

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The potential $V = \chi_{[0,\infty[^n}\varphi$, $\varphi(0) \neq 0$ *always scatters.*

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For any incident wave $u^i \neq 0$ and wavenumber $k > 0$ we have $A_{u^i} \neq 0$.

However if k is a transmission eigenvalue A_{u^i} can become arbitrarily small with $\|u^i\| \geq 1$.

Some follow-up corner scattering results by others

- Päivärinta–Salo–Vesalainen 2017: 2D any angle, 3D almost any spherical cone
- Hu–Salo–Vesalainen 2016: smoothness reduction, new arguments, polygonal scatterer probing
- Elschner–Hu 2015, 2018: 3D any domain having two faces meet at an angle, and also curved edges
- Liu–Xiao 2017: electromagnetic waves
- ...
- free boundary methods:
  - Cakoni–Vogelius 2021: border singularities
  - Salo–Shahgholian 2021: analytic boundary non-scattering
  - ...

Injectivity of the Schiffer's problem for polyhedra

Theorem (Hu–Salo–Vesalainen, Elschner–Hu)

Let P, P' be convex polyhedra and $V = \chi_P \varphi$, $V' = \chi_{P'} \varphi'$ for admissible functions φ, φ'. Then

$$P \neq P' \quad \Longrightarrow \quad A_{u^i} \neq A'_{u^i} \quad \forall u^i \neq 0$$

Any single incident wave determines P in the class of polyhedral penetrable scatterers.

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Ikehata's enclosure method (1999) gives roughly the same!

Stability of polygonal scatterer probing

Non-vanishing total wave

Theorem (B.–Liu 2021)

Let u^i be an incident wave and let $V = \chi_P \varphi$, $V' = \chi_{P'} \varphi'$ be admissible with $|u|, |u'| \neq 0$. Then

$$d_H(P, P') \leq C\left(\ln \ln \|A_{u^i} - A'_{u^i}\|_2^{-1}\right)^{-\eta}$$

for some $\eta > 0$.

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Note 1: stability is still unknown without assuming $|u|, |u'| \neq 0$.
Note 2: is this the optimal stability??

Lower bound for far-field pattern

Arbitrary Herglotz wave

Theorem (B.–Liu 2017)

Let u^i be a normalized Herglotz wave,

$$u^i(x) = \int_{\mathbb{S}^{n-1}} e^{ik\theta\cdot x} g(\theta) d\sigma(\theta), \qquad \|g\|_{L^2(\mathbb{S}^{n-1})} = 1,$$

and let $V = \chi_P \varphi$ be admissible with x_c a corner of P. Then

$$\|A_{u^i}\|_{L^2(\mathbb{S}^{n-1})} \geq C_{\|P_N\|,V} > 0$$

where

$$u^i(x_c + r\theta) = r^N P_N(\theta) + \mathcal{O}(r^{N+1}),$$

$$\|P_N\| = \int_{\mathbb{S}^{n-1}} |P_N(\theta)| \, d\sigma(\theta) > 0.$$

Mistake?

F. Cakoni: “Incident waves that approximate transmission eigenfunctions produce arbitrarily small far-field patterns.”

From apparent contradiction to inspiration

Theorem (B.–Liu 2017)

Let the potential $V = \chi_\Omega \varphi$ be admissible. Let $v, w \neq 0$ be transmission eigenfunctions:

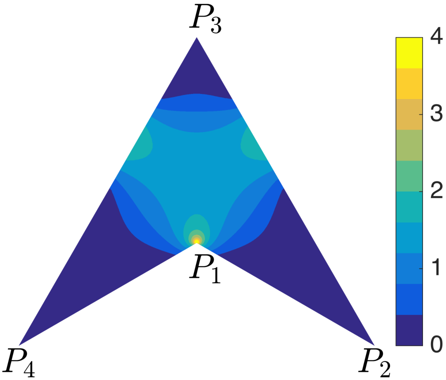
$$
\begin{aligned}
(\Delta + k^2)v &= 0, \qquad \Omega \\
(\Delta + k^2(1+V))w &= 0, \qquad \Omega \\
w - v &\in H_0^2(\Omega).
\end{aligned}
$$

Under C^α-smoothness of v near a convex corner x_c we have

$$v(x_c) = w(x_c) = 0.$$

Transmission eigenfunction localization

B.–Li–Liu–Wang 2017



Piecewise constant determination

Injectivity of piecewise constant potential probing:

Theorem (B., Liu, 2020)

Let Σ_j*,* $j = 1, 2, \ldots$ *be bounded convex polyhedra in an admissible geometric arrangement (think pixels/voxels) and* $V = \sum_j V_j \chi_{\Sigma_j}$*,* $V' = \sum_j V'_j \chi_{\Sigma_j}$ *for constants* $V_j, V'_j \in \mathbb{C}$*. Then*

$$V \neq V' \quad \Longrightarrow \quad A_{u^i} \neq A'_{u^i} \quad \forall u^i(x) = e^{ik\theta\cdot x}$$

if $k > 0$ *small or* $|u| + |u'| \neq 0$ *at each vertex.*

A single incident plane wave determines V in the class of discretized penetrable scatterers if the grid is unknown but same for both V and V'.

Proof sketch

Integration by parts

$$k^2 \int_\Omega (V - V')u' u_0 dx = \int_{\partial\Omega} \big((u - u')\partial_\nu u_0 - u_0 \partial_\nu (u - u')\big) dx$$

if $(\Delta + k^2(1 + V))u_0 = 0$ in Ω.

Proof sketch

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Simple case: $\Omega = B(0, \varepsilon) \cap \Sigma_j$ with $\Sigma_j =]0, 1[^n$

$$\begin{aligned} u'(x) &= u'(0) + u'_r(x) && u' \in H^2 \hookrightarrow C^{1/2} \\ u_0(x) &= e^{\rho \cdot x}(1 + \psi(x)) && \text{CGO} \\ (V - V')(x) &= V_j - V'_j && \text{piecewise constant} \end{aligned}$$

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Hölder estimates give

$$C \left|(V_j - V'_j)u'(0)\right| |\rho|^{-n} = \left|(V_j - V'_j)u'(0) \int_{\mathbb{R}^n_+} e^{\rho \cdot x} dx\right| \leq C |\rho|^{-n-\delta}$$

if $\|\psi\|_p \leq C |\rho|^{-n/p - \varepsilon}$.

Generalizations and limitations

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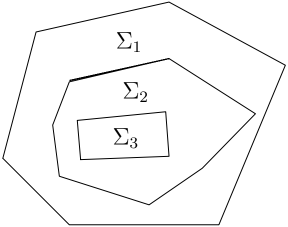

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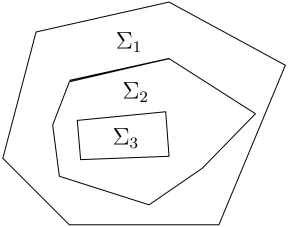
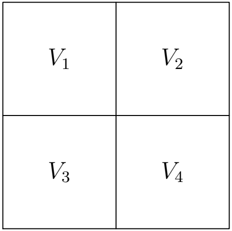
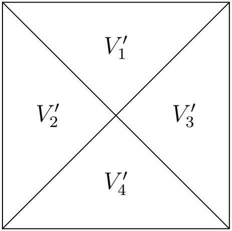


- method cannot yet be shown to distinguish between





Always scattering

High curvature case

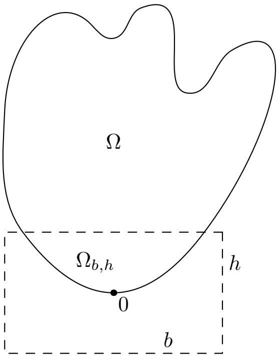
Ω bounded domain, $0 \in \partial\Omega$ admissible K-curvature point.

Theorem (B.–Liu, 2021)

If $f = \chi_\Omega \varphi$, $\varphi \in C^\alpha(\mathbb{R}^n)$ *and*

$$|\varphi(0)| \geq \mathcal{C}(\ln K)^{(n+3)/2} K^{-\delta}$$

then $u_\infty \neq 0$ *for* $(\Delta + k^2)u = f$.



Inverse source problem, Schiffer's problem

$$(\Delta + k^2)u = f = \chi_\Omega \varphi, \qquad \lim_{r\to\infty}(\partial_r - ikr)u = 0$$

Can $u_\infty(\theta) = c\hat{f}(k\theta)$ determine Ω when k is fixed?

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Unique determination:

- $u_\infty = u'_\infty \implies \Omega = \Omega'$ for convex polyhedral shapes (corner scattering). Assuming non-vanishing total waves, also for elasticity (B.–Lin 2018), electromagnetism (B.–Liu–Xiao 2021),
- $u_\infty = u'_\infty \implies \Omega \approx \Omega'$ for convex polyhedral shapes whose corners have been smoothened to admissible K-curvature points (high curvature scattering, B.–Liu 2021),
- $u_\infty = u'_\infty \implies \Omega \approx \Omega'$ for well-separated collections of small scatterers (small source scattering, B.–Liu 2021).

Non-spherical cones

Potential scattering

Let C be any cone whose cross-section K is star-shaped and $\chi_K \in H^\tau(\mathbb{R}^2)$ for some $\tau > 1/2$.

Theorem (B.–Pohjola 2022)

For any $\delta > 0$ there is a cone C_δ such that $d_H(C_\delta, C) < \delta$ and with the following property: potentials of the form

$$V = \chi_{C_\delta}\varphi$$

where φ is smooth enough (roughly $C^{1/4}$) and non-zero at the vertex always scatter.

Non-spherical cones

Source scattering (easier)

Theorem (B.–Pohjola 2022)

A source $f = \chi_C \varphi$ for $(\Delta + k^2)u = f$ scatters for any $k > 0$ when φ is smooth enough and non-zero at the vertex of the cone C when

$$\int_{\mathbb{S}^2 \cap C} Y_2^m dS \neq 0$$

for $m \in \{-2, -1, 0, +1, +2\}$ and Y_2^m is the spherical harmonic of degree 2. This is true if C fits into a thin enough spherical cone.

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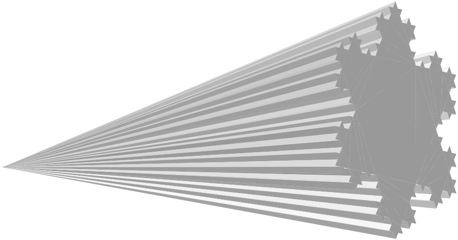
"Thin enough" means $\cos\theta \leq 1/\sqrt{3}$. The magic angle is $\approx 54.74^\circ$.



Scattering screens

A flat screen $\Omega = \Omega_0 \times \{0\}$ with $\Omega_0 \subset \mathbb{R}^2$ simply connected, bounded and smooth. Scattering from such a screen:

$$\begin{aligned} (\Delta + k^2)u^s &= 0, && \mathbb{R}^3 \setminus \overline{\Omega}, \\ u^i + u^s &= 0, && \Omega, \\ r(\partial_r - ik)u^s &\to 0, && r = |x| \to \infty. \end{aligned}$$

Let Ω, Ω' be flat screens, $k > 0$, u^i an arbitary incident wave, and $u^s, u^{s\prime}$ corresponding scattered waves.

Theorem (B.–Päivärinta–Sadique 2020)

- *If* $u^i(x_1, x_2, x_3) + u^i(x_1, x_2, -x_3) \neq 0$ *for some* x *and* $u^s_\infty = {u^{s\prime}}_\infty$ *then* $\Omega = \Omega'$.
- *If* $u^i(x_1, x_2, x_3) + u^i(x_1, x_2, -x_3) = 0$ *for all* x *then* $u^s_\infty = {u^{s\prime}}_\infty = 0$.

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What guarantees vanishing far-fields?

Thank you!